Exercises for Advances in Theoretical Computer Science
Exercise Sheet 6
Due at 06.12.2021, 10:00 s.t.

Remark: For the exercises where you have to define primitive recursive functions, you are allowed to use all functions that were proved to be primitive recursive in the lecture or in a previous exercise.

Exercise 6.1
Prove that the following functions are primitive recursive:

1. $c^k_s : \mathbb{N}^k \rightarrow \mathbb{N}$, where $s \in \mathbb{N}$, defined for every $n \in \mathbb{N}^k$ by: $c^k_s(n) = s$.
2. $\text{fac} : \mathbb{N} \rightarrow \mathbb{N}$, defined for every $n \in \mathbb{N}$ by: $\text{fac}(n) = n!$.
3. $\text{exp} : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, defined for every $(n, m) \in \mathbb{N} \times \mathbb{N}$ by: $\text{exp}(n, m) = n^m$.
4. $\text{eq} : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, defined for every $(n, m) \in \mathbb{N} \times \mathbb{N}$ by: $\text{eq}(n, m) = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$.

Exercise 6.2
Consider the following primitive recursive functions:

I) $f_1 = (+ \circ (+1 \circ 0, \pi_2^2), \star \circ (+1 \circ \pi_1^2, \pi_2^2))$
II) $f_2 = \mathcal{PR}[(+1) \circ 0, \star \circ (+1) \circ \pi_1^2, \pi_2^2)]$
III) $f_3 = \mathcal{PR}[c_1^1, \star \circ (\pi_1^3, \pi_1^3)] \circ (-1 \circ \pi_1^2, \pi_2^2)$

a) Which is the arity of $f_1$, of $f_2$ and of $f_3$? (i.e. how many arguments does each of these functions have?)
b) What do these functions compute if all arguments are equal to 2?
c) What do these functions compute in general?

Exercise 6.3
Express in the form $h(n) = 0$ with $h$ primitive recursive the following conditions:

I) $n$ is between 5 and 12, or $n$ is unequal to $2^m$:
   " $n \geq 5$ and $n \leq 12$, or $n \neq 2^m$ ."

II) $n$ is greater than 20 and is a perfect square:
   " $n > 20$ and $\exists k \leq n : k \cdot k = n$ ."
Exercise 6.4

Prove that the following functions are primitive recursive:

I) $\text{max} : \mathbb{N}^2 \to \mathbb{N}$ defined by $\text{max}(x, y) = \begin{cases} x & \text{if } x \geq y \\ y & \text{otherwise} \end{cases}$.

II) $\text{min} : \mathbb{N}^2 \to \mathbb{N}$ defined by $\text{min}(x, y) = \begin{cases} x & \text{if } x < y \\ y & \text{otherwise} \end{cases}$.

III) $\text{even} : \mathbb{N} \to \mathbb{N}$ defined by $\text{even}(x) = \begin{cases} 1 & \text{if } \exists k \leq x : 2 \ast k = x \\ 0 & \text{otherwise} \end{cases}$.