Remark: In the lecture from 23.12.2021 we sketched a possibility of associating with every Turing Machine $M$ a unique Gödel number $\langle M \rangle \in \mathbb{N}$ such that the coding function and the decoding function are primitive recursive. Similarly, we could associate with every configuration of a given TM a unique Gödel number for the configuration such that coding and decoding are primitive recursive.

The construction uses the following encoding of words as natural numbers: If $\Sigma = \{a_0, a_1, \ldots, a_m\}$ and $w = a_{i_1} \ldots a_{i_n}$ is a word over $\Sigma$ then $\langle w \rangle = (i_1, \ldots, i_n) = \prod_{j=1}^{n} p(j)^{i_j}$. Therefore, we can represent w.l.o.g. words as natural numbers and languages as sets of natural numbers.

Notation: In what follows we will denote by $M_n$ the Turing machine with Gödel number $n$ and with $L(M)$ the language accepted by the Turing machine $M$.

Exercise 9.1

Which functions are computed by:

I) $f_1 = \mu c_1^2$

II) $f_2 = \mu g$, where $g(n, i) = \begin{cases} n + 1 & \text{if } i = 0 \\ \mu j(j + 1 + n = 0) & \text{if } i = 1 \\ 0 & \text{if } i \geq 2 \end{cases}$

III) $f_3 = \mu g$, where $g(n, i) = \begin{cases} n + 1 & \text{if } i = 0 \\ \mu j((j + 1) - n = 0) & \text{if } i = 1 \\ 0 & \text{if } i \geq 2 \end{cases}$

Exercise 9.2

Consider the definition of the Ackermann function given in the lecture:

$$A(0, y) = y + 1$$
$$A(x + 1, 0) = A(x, 1)$$
$$A(x + 1, y + 1) = A(x, A(x + 1, y))$$

$\text{Ack}(x) = A(x, x)$

For every $m \in \mathbb{N}$, let

$$B_m = \{ f \mid f \text{ is primitive recursive and for all } n_1, \ldots, n_r \in \mathbb{N} \text{ where } r \text{ is the arity of } f \text{ it is } f(n_1, \ldots, n_r) < A(m, \sum_{i=1}^{r} n_i) \}$$

We assume that the following properties of the function $A$ are known (proving these facts is not part of this exercise).
(1) \( A(1, y) = y + 2 \) for every \( y \in \mathbb{N} \)
(2) \( A(2, y) > 2y \) for every \( y \in \mathbb{N} \)
(3) \( A(3, y) > 2^{y+1} \) for every \( y \in \mathbb{N} \)
(4) \( y < A(x, y) \) for all \( x, y \in \mathbb{N} \)
(5) \( A(x, y) < A(x, y+1) \) for all \( x, y \in \mathbb{N} \)
(6) \( A(x, y+1) \leq A(x+1, y) \) for all \( x, y \in \mathbb{N} \)
(7) \( A(x, y) < A(x+1, y) \) for all \( x, y \in \mathbb{N} \).
(8) \( A(x, 2y) < A(x+3, y) \) for all \( x, y \in \mathbb{N} \).
(9) If \( f, g_1, \ldots, g_r \in B_m \) and \( h = f \circ (g_1, \ldots, g_r) \) then there exists a natural number \( m' \) (depending on \( m \) and \( r \)) such that \( h \in B_{m'} \).
(10) If \( g, h \in B_m \) and \( f \) is defined by primitive recursion from \( g \) and \( h \) then \( f \in B_{m+4} \).

Prove (possibly using some of the information above) that:

I) \( 0 < A(0,0) \) for all \( n \in \mathbb{N} \)
II) \( \pi^*_1(n_1, \ldots, n_r) < A(0, \sum_{i=1}^r n_i) \) for all \( n_1, \ldots, n_r \in \mathbb{N} \)
III) \( n+1 < A(1, n) \) for all \( n \in \mathbb{N} \)
IV) For every primitive recursive function \( f \) there exists \( m \in \mathbb{N} \) with \( f \in B_m \).
V) The Ackermann function \( \text{Ack} \) defined by \( \text{Ack}(n) = A(n, n) \) is not primitive recursive.

**Exercise 9.3**

Let \( K = \{ n \mid M_n \text{ halts on input } n \} \).

a) Prove that \( K \) is undecidable.

b) Prove that \( K \) is acceptable.

c) Prove that the complement of \( K \) is not acceptable.

**Exercise 9.4**

We define the following relation \( \leq \) on languages (regarded as sets of natural numbers):

If \( L_1, L_2 \) are two languages (regarded as sets of natural numbers), we say that \( L_1 \leq L_2 \) if there exists a TM computable function \( f : \mathbb{N} \to \mathbb{N} \) with the property that:

\[
\forall n \in \mathbb{N} \quad n \in L_1 \text{ if and only if } f(n) \in L_2.
\]

Prove that the relation \( \leq \) is transitive, i.e. that if \( L_1, L_2 \) and \( L_3 \) are languages (regarded here as sets of natural numbers) such that \( L_1 \leq L_2 \) and \( L_2 \leq L_3 \), then \( L_1 \leq L_3 \).

\[\text{If you want to submit solutions, please do so until 03.01.2022, 10:00 s.t. via e-mail (with “Homework ACTCS” in the subject) to dpeuter@uni-koblenz.de.}\]