Detailed Example for a Reduction

\( P := \{ n \mid L(M_n) \text{ is finite} \} \)

We show that \( P \) is undecidable by reducing the complement of \( H_0 \) to \( P \), i.e. proving that \( \overline{H_0} \leq P \), where \( \overline{H_0} := \{ n \mid M_n \text{ does not halt for input 0} \} \).

We have to find a Turing machine-computable function \( f : \mathbb{N} \to \mathbb{N} \) such that \( i \in \overline{H_0} \iff f(i) \in P \).

Let \( i \in \mathbb{N} \), and let \( M_i \) be the Turing machine with Gödel number \( i \).

We construct \( M_j := R(M_i) \), i.e. \( M_j \) simulates \( M_i \) on tape 2 with empty input. For all \( i \in \mathbb{N} \) we define \( f(i) = j \) as above.

If \( M_i \) does not halt on input 0, \( M_j \) halts on no input and thus accepts \( \emptyset \), which is a finite language. If \( M_i \) halts on input 0, \( M_j \) halts on all inputs and thus accepts \( \Sigma^* \), which is an infinite language.

\[
L(M_j) = \begin{cases} 
\emptyset & \text{if } M_i \text{ does not halt on input 0} \\
\Sigma^* & \text{if } M_i \text{ halts on input 0}
\end{cases}
\tag{1}
\]

Then:

\[
\begin{align*}
& f(i) = j \in P \\
\iff & L(M_j) \text{ is finite} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (\text{by the definition of } P) \\
\iff & L(M_j) = \emptyset \\
\iff & L(M_j) \text{ can be either } \emptyset \text{ or } \Sigma^*, \text{ but } \Sigma^* \text{ is infinite} \\
\iff & M_i \text{ does not halt on input 0} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{(by (1))} \\
\iff & i \in \overline{H_0} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{(by the definition of } \overline{H_0})
\end{align*}
\]
The G"odel number of $M_{i_1}^{(2)}$. As $i_1 \in H_0$, $M_{i_1}$ does not halt on 0, so $L(M_{j_1}) = \emptyset$, hence $j_1 \in P$.

The G"odel number of $M_{i_2}^{(2)}$. As $i_2 \in H_0$, $M_{i_2}$ halts on 0, so $L(M_{j_2}) = \Sigma^*$, hence $j_2 \notin P$.

Figure 1:

- $\mathbb{N} = H_0 \cup \overline{H}_0$
- $\mathbb{N} = P \cup \overline{P}$
- $J = \{f(n) \mid n \in \mathbb{N} \text{ and } f \text{ defined as above}\}$

$\mathbb{N}$ is the set of all G"odel numbers. On the left hand side it is shown as the union of the two disjoint sets $H_0$ and $\overline{H}_0$. On the right hand side it is shown as the union of the two disjoint sets $P$ and $\overline{P}$. $J$ is the range of the function $f$ defined above. For all $i \in H_0$, $f(i) \in \overline{P} \cap J$ and for all $i \in \overline{H}_0$, $f(i) \in P \cap J$. 