1. The technique of forcing
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Motivation

Theorem (Cantor)

There is no surjection from $\mathbb{N}$ onto $\mathcal{P}(\mathbb{N})$.

This lead to the study of infinite cardinals.

Definition

- $\aleph_0$ denotes the cardinality of $\omega = \mathbb{N}$.
- $\aleph_1$ denotes the least uncountable cardinal.
- $\aleph_2$ denotes the second-least uncountable cardinal.
- $c = 2^{\aleph_0}$ denotes the cardinality of $\mathcal{P}(\omega)$.
In 1878, Georg Cantor conjectured the Continuum Hypothesis:

**Conjecture (The Continuum Hypothesis (CH))**

There is no cardinal between $\aleph_0$ and $2^{\aleph_0}$, i.e. $2^{\aleph_0} = \aleph_1$.

- The Continuum Hypothesis was the first of Hilbert’s famous list of 23 open Problems.
- In 1940, Kurt Gödel constructed a model of ZFC (denoted L) in which the Continuum Hypothesis holds.
- In 1963, Paul Cohen proved that CH is actually independent of the axioms of ZFC, i.e. both ZFC + CH and ZFC + ¬CH are consistent.

The method he used (and developed for this result) is forcing.
A forcing (notion) is a non-atomic partial order $\mathbb{P} = (P, \leq_P, 1_P)$ with a maximal element $1_P$, i.e.

- $\leq_P$ is reflexive, transitive and antisymmetric,
- for every $p \in P$, $p \leq_P 1_P$,
- for every $p \in P$ there are $p_0, p_1 \leq_P p$ which are incompatible, i.e. there is no $r \in P$ with $r \leq_P p_0, p_1$. 
The technique of forcing

Forcing notions

Example (Cohen forcing)

Let $\mathbb{P} = \text{Fn}(\omega, 2, \aleph_0)$ be the set of partial functions $p : \text{dom}(p) \to 2$ with $\text{dom}(p) \subseteq \omega$ finite, ordered by reverse inclusion, i.e.

$$p \leq_{\mathbb{P}} q \iff \text{dom}(p) \supseteq \text{dom}(q) \text{ and } p \upharpoonright \text{dom}(q) = q.$$ 

Furthermore, let $1_{\mathbb{P}} = \emptyset$, the empty function.
Generic filters

Definition

Let $M$ be a countable transitive model of ZFC and $\mathbb{P} \in M$ be a forcing notion.

1. A subset $D \subseteq \mathbb{P}$ is said to be dense, if for every $p \in P$ there is some $q \leq_P p$ with $q \in D$.

2. A subset $G \subseteq \mathbb{P}$ is said to be a $\mathbb{P}$-generic filter, if it has the following properties:
   - If $p \leq_P q$ and $p \in G$, then $q \in G$.
   - If $p, q \in G$ then there is $r \in G$ such that $r \leq_P p, q$.
   - If $D \subseteq \mathbb{P}$ is a dense set which is in $M$, then $G \cap D \neq \emptyset$.

Example

Consider Cohen forcing $\mathbb{P} = \text{Fn}(\omega, 2, \aleph_0)$. Then for each $n \in \omega$, the set $D_n = \{p \in \mathbb{P} \mid n \in \text{dom}(p)\}$ is dense.
Names

**Definition**

Let $\mathbb{P}$ be a forcing notion. A $\mathbb{P}$-name is a set whose elements are of the form $(\sigma, p)$, where $\sigma$ is a $\mathbb{P}$-name and $p \in \mathbb{P}$.

This definition requires *transfinite recursion*.

**Example**

Let $M \models \text{ZFC}$ and $\mathbb{P} \in M$ a forcing notion.

- $\emptyset$ is a $\mathbb{P}$-name.
- Let $x \in M$. Then there is a *canonical* $\mathbb{P}$-name for $x$ given by
  \[
  \check{x} = \{ (\check{y}, 1_\mathbb{P}) \mid y \in x \}.
  \]
- The set $\dot{G} = \{(\check{p}, p) \mid p \in \mathbb{P}\}$ is a $\mathbb{P}$-name, the canonical $\mathbb{P}$-name for a $\mathbb{P}$-generic filter.
The technique of forcing

Evaluations of names

Let $\mathbb{P}$ be a forcing notion and $G$ a $\mathbb{P}$-generic filter. We can evaluate a $\mathbb{P}$-name $\sigma$ as follows:

$$\sigma^G = \{\tau^G | \exists p \in G : (\tau, p) \in \sigma\}.$$

Example

- $\emptyset^G = \emptyset$.
- Let $x \in M$. Then
  $$\check{x}^G = \{\check{y}^G | y \in x\} = \{y | y \in x\} = x$$
  by transfinite induction.
- $\dot{G}^G = \{\check{p}^G | p \in G\} = G$. 

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Generic extensions

Let $M \models \text{ZFC}$, $\mathbb{P} \in M$ be a non-trivial forcing notion and $G$ a $\mathbb{P}$-generic filter. Then we define

$$M[G] = \{\sigma^G \mid \sigma \text{ is a } \mathbb{P}\text{-name}\}.$$  

Fact

1. $M \cup \{G\} \subseteq M[G]$
2. $G \notin M$.

Proof.

For (2) suppose that $G \in M$. Then the set $D = \mathbb{P} \setminus G$ is in $M$. Moreover, $D$ is dense: Let $p \in \mathbb{P}$. Since $\mathbb{P}$ is non-atomic, there are $p_0, p_1 \leq \mathbb{P} p$ such that $p_0$ and $p_1$ are incompatible. But then at least one of $p_0, p_1$ is not in $G$. 

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The technique of forcing

Where do generic filters live?

Suppose that $V \models \text{ZFC}$ and $V$ contains countable, transitive models $M \in V$ of ZFC. Let $\mathbb{P} \in M$ be a forcing notion. We extend $M$ to $M[G]$, where $G$ is a $\mathbb{P}$-generic filter contained in $V$.

**Fact**

*For every $p \in \mathbb{P}$ there is a $\mathbb{P}$-generic filter $G$ in $V$ with $p \in G$.**

**Proof.**

Since $M$ is countable, $M$ contains only countably many dense subsets of $\mathbb{P}$. Let $(D_n \mid n \in \omega)$ enumerate them (in $V$). Now we inductively construct a sequence of conditions $(p_n \mid n \in \omega)$ by

- $p_0 = p$
- Given $p_n$, let $p_{n+1} \leq \mathbb{P} p_n$ with $p_{n+1} \in D_n$.

Then $G = \{ q \in \mathbb{P} \mid \exists n \in \omega : p_n \leq \mathbb{P} q \}$ is a $\mathbb{P}$-generic filter over $M$.  

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The idea behind names

Every element of $M[G]$ has a $\mathbb{P}$-name $\sigma \in M$ but $M$ does not know how $\sigma$ will be evaluated. This is similar to the case of extensions of fields:

Consider $\mathbb{Q}$ and an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$.

- In $\mathbb{Q}$, the polynomial $X^2 - 2$ names the root $\sqrt{2} \in \overline{\mathbb{Q}}$.
- If we extend $\mathbb{Q}$ to $\mathbb{Q}[\sqrt{2}]$, then $\sqrt{2}$ is the evaluation of $X^2 - 2$. 
Cohen forcing

Let $\mathbb{P}$ denote $Fn(\omega, 2, \aleph_0)$ and let $G$ be $\mathbb{P}$-generic over $M$. In $M[G]$, consider the function

$$c = \bigcup_{p \in G} p. \quad (c \text{ is called a Cohen real})$$

**Fact**

1. $c$ is a function $\omega \rightarrow 2$.
2. If $f \in M$ is a function $f : \omega \rightarrow 2$ then $f \neq c$.

**Proof.**

(1)

- Let $n \in \omega$. If there are $p, q \in G$ such that $n \in \text{dom}(p) \cap \text{dom}(q)$ and $p(n) \neq q(n)$, then $p$ and $q$ are incompatible.

- To see that $\text{dom}(c) = \omega$, note that $D_n = \{p \in \mathbb{P} \mid n \in \text{dom}(p)\}$ is dense. Let $p \in G \cap D_n$. Then $n \in \text{dom}(c)$ and $c(n) = p(n)$.
Cohen forcing

Let $P$ denote $Fn(\omega, 2, \aleph_0)$ and let $G$ be $P$-generic over $M$. In $M[G]$, consider the function

$$c = \bigcup_{p \in G} p.$$ (c is called a Cohen real)

**Fact**

1. $c$ is a function $\omega \to 2$.
2. If $f \in M$ is a function $f : \omega \to 2$ then $f \neq c$.

**Proof.**

(2) Consider $D_f = \{p \in P \mid \exists n \in \text{dom}(p) : p(n) \neq f(n)\}$. We claim that $D_f$ is a dense subset of $P$. Let $p \in P$. Since $\text{dom}(p)$ is finite, there is $n \in \omega \setminus \text{dom}(p)$. Then $q = p \cup \{(n, 1 - f(n))\} \leq_P p$ and $q \in D_f$.

Take $p_0 \in G \cap D_f$ and let $n_0 \in \text{dom}(p_0)$ such that $p_0(n_0) \neq f(n_0)$. Then $c(n_0) = p_0(n_0)$ and so $c \neq f$. 

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Theorem (Cantor)

There is no surjection from \( \omega \) onto \( \mathcal{P}(\omega) \).

Proof.

Suppose the contrary. Observe that \( \mathbb{P} = Fn(\omega, 2, \aleph_0) \) is countable. Then so is \( \mathcal{P}(\mathbb{P}) \), so we can enumerate (in \( M \)) all dense subsets of \( \mathbb{P} \). But then we can construct in \( M \) a \( \mathbb{P} \)-generic filter over \( M \).
We identify $\mathbb{R}$ with $\omega_2$, the space of functions $\omega \to 2$.

Observation

If $\mathbb{P}$ is a forcing notion and $G$ is $\mathbb{P}$-generic over $M$, then $\mathbb{R}^M$ and $\mathbb{R}^{M[G]}$ do not have to be the same.
We define intervals in the following way: For 
\( s \in <\omega 2 = \{ t : \text{dom}(t) \to 2 \mid \text{dom}(t) \in \omega \} \) we define

\[
I_s = \{ x \in \mathbb{R} \mid s \subseteq x \} \subseteq \mathbb{R}
\]

and we define \( \mu(I_s) = 2^{-\text{dom}(s)}. \)

**Definition**

We say that a set \( X \subseteq \mathbb{R} \) has *measure zero*, if for every \( \varepsilon > 0 \) there exists a sequence \( (I_n \mid n \in \omega) \) of intervals in \( \mathbb{R} \) such that \( X \subseteq \bigcup_{n \in \omega} I_n \) and 
\[
\sum_{n \in \omega} \mu(I_s) < \varepsilon.
\]
Theorem

Let $M[G]$ be a generic extension of $M$ by Cohen forcing, and let $c \in M[G]$ be the canonical Cohen real. Then in $M[G]$, $\mathbb{R}^M$ has measure zero.

Proof.

Let $\varepsilon > 0$, let $k \in \omega$ such that $\varepsilon > 2^{-k}$. For $n \in \omega$ define

$$s_n : k + n + 1 \to 2, \quad s_n(l) = c(n + l).$$

Now define $l_n = l_{s_n}$. Then

$$\sum_{n \in \omega} \mu(l_n) = \sum_{n \in \omega} 2^{-(k+n+1)} = 2^{-k} < \varepsilon.$$
Proof, continued.

$s_n : k + n + 1 \to 2$, $s_n(l) = c(n + l)$, $l_n = l_{s_n}$.

It remains to check: $\mathbb{R}^M \subseteq \bigcup_{n \in \omega} l_n$. Let $x \in \mathbb{R}^M$. Then

$$D_x = \{p \in \mathbb{P} \mid \exists n \in \omega \forall l < k + n + 1 : n + l \in \text{dom}(p) \wedge p(n + l) = x(l)\} \in M$$

is a dense subset of Cohen forcing $\mathbb{P}$. Take $p \in G \cap D_x$ and $n \in \omega$ which witnesses that $p \in D_x$. Then

$$\forall l < k + n + 1 : s_n(l) = c(n + l) = p(n + l) = x(l)$$

and so $s_n \subseteq x$ and $x \in l_n$. 

\qed
We have used Cohen forcing to add one real $c$ which is not an element of the ground model $M$. What happens if we add $\aleph_2$-many Cohen reals in this way?

**Definition**

Let $F\mathfrak{n}(\mathfrak{N}_2^M \times \omega, 2, \mathfrak{N}_0)$ denote the set of finite partial functions from $\mathfrak{N}_2^M \times \omega \to 2$, ordered by reverse inclusion.
Let $\mathbb{P}$ denote $\text{Fn}(\aleph_2^M \times \omega, 2, \aleph_0)$ and let $G$ be $\mathbb{P}$-generic over $M$.

**Lemma**

The following statements hold:

1. $F = \bigcup_{p \in G} p$ is a function $F : \aleph_2^M \times \omega \to 2$.

2. For each $\alpha < \aleph_2^M$, let $c_\alpha(n) = F(\alpha, n)$. Then for all $\alpha < \beta < \aleph_2^M$, $c_\alpha \neq c_\beta$.

**Proof.**

(1) Follows from the density of the sets

$$D_{\alpha,n} = \{ p \in \mathbb{P} \mid (\alpha, n) \in \text{dom}(p) \}$$

for all $\alpha < \aleph_2^M$ and $n \in \omega$. \qed
Let $\mathbb{P}$ denote $\text{Fn}(\mathbb{N}_2^M \times \omega, 2, \mathbb{N}_0)$ and let $G$ be $\mathbb{P}$-generic over $M$.

**Lemma**

The following statements hold:

1. $F = \bigcup_{p \in G} p$ is a function $F : \mathbb{N}_2^M \times \omega \to 2$.
2. For $\alpha < \mathbb{N}_2^M$, let $c_\alpha(n) = F(\alpha, n)$. Then for all $\alpha < \beta < \mathbb{N}_2^M$, $c_\alpha \neq c_\beta$.

**Proof.**

(2) For all $\alpha < \beta < \mathbb{N}_2^M$ consider

$$D_{\alpha\beta} = \{ p \in \mathbb{P} \mid \exists n \in \omega : (\alpha, n), (\beta, n) \in \text{dom}(p) \text{ and } p(\alpha, n) \neq p(\beta, n) \}.$$  

Then $D_{\alpha\beta}$ is dense in $\mathbb{P}$. By genericity, take $p \in G \cap D_{\alpha\beta}$ and $n \in \omega$ such that $(\alpha, n), (\beta, n) \in \text{dom}(p)$ with $p(\alpha, n) \neq p(\beta, n)$. Then

$$c_\alpha(n) = F(\alpha, n) = p(\alpha, n) \neq p(\beta, n) = F(\beta, n) = c_\beta(n).$$
The previous lemma shows that there are at least \( \aleph_2^M \) many reals in \( M[G] \), so if we can show that \( \aleph_2^M = \aleph_2^{M[G]} \) then in \( M[G] \)

\[
(2^{\aleph_0})^M[G] = |\mathbb{R}^M[G]| \geq \aleph_2^M = \aleph_2^{M[G]}.
\]

**Definition**

Let \( \mathbb{P} \) be a forcing notion.

1. An *antichain* in \( \mathbb{P} \) is a subset \( A \subseteq \mathbb{P} \) with the property that all elements of \( A \) are incompatible, i.e. for all \( p, q \in A \), there is no \( r \in \mathbb{P} \) with \( r \leq \mathbb{P} p, q \).

2. \( \mathbb{P} \) satisfies the *countable chain condition (ccc)*, if every antichain in \( \mathbb{P} \) is at most countable.
Theorem

If $\mathbb{P}$ satisfies the ccc then $\mathbb{P}$ preserves all cardinals, i.e. $|\kappa|^{M[G]} = \kappa$ in every $\mathbb{P}$-generic extension $M[G]$. In particular, $\aleph_2^{M[G]} = \aleph_2^M$.

How can it happen that $\aleph_2^M$ is collapsed?
Forcing $\neg \text{CH}$

**Theorem**

If $\mathbb{P}$ satisfies the ccc then $\mathbb{P}$ preserves all cardinals, i.e. $|\kappa|^{M[G]} = \kappa$ in every $\mathbb{P}$-generic extension $M[G]$. In particular, $\aleph_2^{M[G]} = \aleph_2^M$.

**Example**

Consider the forcing notion

$$\text{Col}(\omega, \aleph_2^M) = \{ p : \text{dom}(p) \to \aleph_2^M \mid \text{dom}(p) \subseteq \omega \text{ finite} \}$$

ordered by reverse inclusion. If $G$ is $\text{Col}(\omega, \aleph_2^M)$-generic, then $F = \bigcup_{p \in G} p$ is a function $\omega \to \aleph_2^M$. Then $F$ is surjective because for each $\alpha < \aleph_2^M$ the set

$$D_\alpha = \{ p \in \text{Col}(\omega, \aleph_2^M) \mid \exists n \in \text{dom}(p) : p(n) = \alpha \}$$

is dense. So in $M[G]$ we have $\aleph_2^M = \aleph_0^{M[G]}$.
Lemma

\[ \mathbb{P} = \text{Fn}(\aleph_2^M \times \omega, 2, \aleph_0) \text{ has the ccc.} \]

Proof.

Suppose that \((p_i \mid i < \aleph_1^M)\) enumerates an uncountable antichain. Then there is a set \(X \subseteq \aleph_1^M\) of size \(\aleph_1^M\) and \(n \in \omega\) such that

\[ \forall i \in X : |\text{dom}(p_i)| = n. \]

Now let \(m \leq n\) be maximal such that there is a set \(b\) with \(|b| = m\) and \(Y \subseteq X\) of size \(\aleph_1^M\) such that for all \(i \in Y\), \(b \subseteq \text{dom}(p_i)\).

We claim that for all \(x \notin b\) there is \(i(x) < \aleph_1^M\) such that for all \(i > i(x)\) in \(Y\), \(x \notin \text{dom}(p_i)\). This holds because otherwise \(b \cup \{x\}\) would contradict the maximality of \(m\).
Proof, continued.

- $Y \subseteq \aleph_1$ of size $\aleph_1^M$ with $b \subseteq \text{dom}(p_i)$ for all $i \in Y$.
- for all $x \notin b$, $i(x) < \aleph_1^M$ with $x \notin \text{dom}(p_i)$ for all $i > i(x)$ in $Y$.

We construct an increasing sequence $(i_\xi \mid \xi < \aleph_1)$ with $i_\xi \in Y$ by recursion:

Suppose that $(i_\xi \mid \xi < \zeta)$ has already been defined. Then let $i_\zeta$ be the minimal ordinal $i \in Y$ such that

$$i > \sup\{ i(x) \mid \exists \xi < \zeta : x \in \text{dom}(p_\xi) \}.$$

Now let $\xi < \zeta < \aleph_1^M$. We claim that $\text{dom}(p_{i_\xi}) \cap \text{dom}(p_{i_\zeta}) \subseteq b$. Let $x \in \text{dom}(p_{i_\xi}) \cap \text{dom}(p_{i_\zeta})$ and $x \notin b$. Then $i_\zeta > i(x)$ and so $x \notin \text{dom}(p_{i_\zeta})$.

But there are only countably many possibilities for $p_{i_\xi} \upharpoonright b$, so there must be some $\xi \neq \zeta$ such that $p_{i_\xi}$ and $p_{i_\zeta}$ are compatible.
Thus we have proved

Theorem (Gödel, Cohen)

\[ \text{CH is independent of the axioms of ZFC.} \]

Thank you for your attention!