

The expression lemma^{*}

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Abstract. Algebraic data types and catamorphisms (folds) play a central role in functional programming as they allow programmers to define recursive data structures and operations on them uniformly by structural recursion. Likewise, in object-oriented (OO) programming, recursive hierarchies of object types with virtual methods play a central role for the same reason. There is a semantical correspondence between these two situations which we reveal and formalize categorically. To this end, we assume a coalgebraic model of OO programming with functional objects. The development may be helpful in deriving refactorings that turn sufficiently disciplined functional programs into OO programs of a designated shape and vice versa.

Key words: expression lemma, expression problem, functional object, catamorphism, fold, the composite design pattern, program calculation, distributive law, free monad, cofree comonad.

1 Introduction

There is a folk theorem that goes as follows. Given is a recursively defined data structure with variants d_1, \dots, d_q , and operations o_1, \dots, o_r that are defined by structural recursion on the data variants. There are two equivalent implementations. In the *functional* style, we define recursive functions f_1, \dots, f_r such that each f_i implements o_i and is defined by q equations, one equation for each d_j . In the *object-oriented* (OO) style, we define object types t_1, \dots, t_q such that each t_j uses d_j as its opaque state type, and it implements a common interface consisting of methods m_1, \dots, m_r with the types of f_1, \dots, f_r , except that the position of structural recursion is mapped to “self” (say, “this”). The per-variant equations of the functional style correspond to the per-variant method implementations of the OO style.

This folk theorem is related to the so-called *expression problem* [40,43,34,24,23], which focuses on the extensibility trade-offs of the different programming styles, and aims at improved language designs with the best possible extensibility for all possible scenarios. Such a comparison of styles can definitely benefit from an understanding of the semantical correspondence between the styles, which is indeed the overall contribution of the present paper. We coin the term *expression lemma* to refer to the sketched

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functional/OO correspondence. We do not discuss the expression problem any further in this paper, but we contend that the expression lemma backs up past and future work on the expression problem. It is also assumed that the lemma contributes to the foundation that is needed for future work on refactorings between the aforementioned styles, e.g., in the context of making imperative OO programs more pure, more functional, or more parallelizable.

Contributions

The paper provides a technical formulation of the expression lemma, in fact, the first such formulation, as far as we know. The formulation is based on comparing functional programs (in particular, folds) with coalgebraically modeled OO programs. We provide first ever, be it partial answers to the following questions:

- When is a functional program that is defined by recursive functions on an algebraic data type semantically equivalent to an OO program that is defined by recursive methods on object structures?
- What is the underlying common definition of the two programs?

The formal development is done categorically, and essentially relies on the established concept of distributive laws of a functor over a functor and (co)monadic generalizations thereof. A class of pairs of functional and OO programs that are duals of each other is thus revealed and formalized. This makes an important contribution to the formal understanding of the semantical equivalence of functional and OO programming. Nontrivial facets of dualizable programs are identified — including facets for the freewheeling use of term construction (say, object construction) and recursive function application (say, method invocation).¹

Road-map

The paper is organized as follows. §2 starts off by providing a simple pair of functional and OO programs that we use as a motivating example. §3 sketches (a simple version of) the expression lemma and its proof. §4 approaches the expression lemma categorically. §5 interprets the basic formal development, and suggests extensions to cover a larger class of dualizable programs. §6 formalizes the proposed extensions. §7 discusses related work. §8 concludes the paper.

2 A motivating example

Consider the problem of modeling a representation of arithmetic expressions such as numeric literals and binary expressions for addition. In OO programming, say in the Java language, we devise a class hierarchy as follows; there is one class for literals, and another class for addition:

¹ The paper comes with a source distribution that contains a superset of all illustrations in the paper as well as a Haskell library that can be used to code dualizable programs. Refer to the paper's web site.

```

public abstract class Expr { }
public class Num extends Expr { public int value; }
public class Add extends Expr { public Expr left, right; }

```

In functional programming, say in the Haskell language, expressions may be modeled as an algebraic data type as follows; there is one data-type constructor for literals, and another constructor for additions:

```

-- Arithmetic expression forms
data Expr = Num Int | Add Expr Expr

```

The direct correspondence of these two codes may be obvious enough, but we must not stop here. The object model is atypical in so far that the field declarations (for `value`, `left`, and `right`) are *public*, and the classes do not seem to implement any interesting interface. Let us consider an object model with operations; we list two such operations as abstract methods in the abstract base class for expressions:

```

public abstract class Expr {
    public abstract int eval (); // Evaluate the expression
    public abstract void modn(int v); // Modify literals modulo v
}

```

(We could use Java interfaces as well.) The concrete subclasses implement the abstract methods in the obvious way; they also provide constructors to initialize the state of newly constructed objects. Thus:

```

public abstract class Expr {
    public abstract int eval ();
    public abstract void modn(int v);
}

public class Num extends Expr {
    private int value;
    public Num(int value) { this.value = value; }
    public int eval () { return value; }
    public void modn(int v) { this.value = this.value % v; }
}

public class Add extends Expr {
    private Expr left, right;
    public Add(Expr left, Expr right) { this.left = left; this.right = right; }
    public int eval () { return left.eval () + right.eval (); }
    public void modn(int v) { left.modn(v); right.modn(v); }
}

```

It is worth emphasizing that the method implementations in the `Add` class adopt the common style of operations in the *composite pattern* of OO programming [14]. That is, to implement `eval` and `modn` for the composite case, i.e., for `Add` expressions, the operations are invoked for the components (`left` and `right`), and possibly the intermediate results are combined into a compound result (which is the case for the `eval` operation).

In functional programming, we define `eval` and `modn` as recursive functions:

```
-- Evaluate expressions
eval :: Expr -> Int
eval (Num i) = i
eval (Add l r) = eval l + eval r

-- Modify literals modulo v
modn :: Expr -> Int -> Expr
modn (Num i) v = Num (i `mod` v)
modn (Add l r) v = Add (modn l v) (modn r v)
```

Intuitively, we can “see” that both models are functionally equivalent, i.e., these are two views on the same conceptual data structure of arithmetic expressions with *constructors*, *observers* (such as `eval`), and *modifiers* (such as `modn`). However:

We seek a proved, formal statement of semantical equivalence.

In the process of making this contribution, we will also clarify the class of functional and OO programs that are covered by the alleged correspondence. As a first approximation, we quantify as follows: functions that can be defined by folds of a certain disciplined kind are amenable to the dualization to OO programs.

3 Informal development

In order to clarify the correspondence between the functional and the OO style, we need a setup that admits the comparison of both kinds of programs. In particular, we must introduce a suitable formalism for objects. We adopt the coalgebraic view of functional objects, where object interfaces are modeled as *interface endofunctors*, and implementations are modeled as *coalgebras* of these functors. We will explain the essential concepts here, but refer to [31,18] for a proper introduction to the subject, and to [30] for a type-theoretical view. In the present section, we illustrate the coalgebraic model in Haskell, while the next section adopts a more rigorous, categorical approach.

3.1 Interfaces and coalgebras

The following type constructor models the *interface* of the base class `Expr`:

```
type IExprF x = (Int, Int -> x)
```

Here, `x` is the type parameter for the object type that implements the interface; the first projection corresponds to the observer `eval` (and hence its type is `Int`); the second projection corresponds to the modifier `modn` (and hence its type is `Int -> x`, i.e., the method takes an `Int` and returns a new object of the same type `x`). We also need a type that hides the precise object type — in particular, its state representation; this type would effectively be used as a bound for all objects that implement the interface. To this end, we may use the recursive closure of `IExprF`:

<pre>-- eval function for literals numEval :: Int → Int numEval = id</pre>	<pre>-- modn function for literals numModn :: Int → Int numModn = mod</pre>
<pre>-- eval function for additions addEval :: (Int, Int) → Int addEval = uncurry (+)</pre>	<pre>-- modn function for addition addModn :: (Int→a, Int→a)→Int→(a,a) addModn = uncurry (/\<)</pre>

Fig. 1. Component functions of the motivating example.

```
newtype IExpr = InIExpr { outIExpr :: IExprF IExpr }
```

Method calls boil down to the following convenience projections:

```
call_eval = fst . outIExpr
call_modn = snd . outIExpr
```

Implementations of the interface are *coalgebras* of the IExprF functor. (The notion of coalgebra, essentially, only means here that the type is of the shape $x \rightarrow f x$, while f is a functor. An interface functor forms a product at the top-level — one projection for each method.) Thus, the type of IExprF implementations is the following:

```
type IExprCoalg x = x → IExprF x
```

Here are the straightforward implementations for the Expr hierarchy:

```
numCoalg :: IExprCoalg Int
numCoalg = numEval /\ numModn

addCoalg :: IExprCoalg (IExpr, IExpr)
addCoalg = (addEval . (call_eval <*> call_eval)) /\
           (addModn . (call_modn <*> call_modn))
```

For clarity (and reuse), we have factored out the actual functionality per function and data variant into helper functions numEval, addEval, numModn, and addModn; c.f. Fig. 1.² Note that the remaining polymorphism of addModn is essential to maintain enough naturality needed for our construction.

The above types clarify that the implementation for literals uses int as the *state type*, whereas the implementation for additions uses (IExpr, IExpr) as the state type. Just as the original Java code invoked methods on the components stored in left and right, the coalgebraic code applies the corresponding projections of the IExpr-typed values call_eval and call_modn. It is important to keep in mind that IExpr-typed values

² We use point-free (pointless) notation (also known as Backus' functional forms) throughout the paper to ease the categorical development. In particular, we use these folklore operations: $f <*> g$ maps the two argument functions over the components of a pair; $f /\ g$ constructs a pair by applying the two argument functions to the same input; $f <|> g$ maps over a sum with an argument function for each case; $f \setminus g$ performs case discrimination on a sum with an argument function for each case. Refer to Fig. 2 for a summary of the used operations.

```

-- Function composition (defined in the Haskell Prelude)
(.) :: (b -> c) -> (a -> b) -> a -> c
(f . g) x = f (g x)

-- Projections, currying, uncurrying for products (defined in the Haskell Prelude)

fst :: (a,b) -> a
fst (x,y) = x

snd :: (a,b) -> b
snd (x,y) = y

curry :: ((a, b) -> c) -> a -> b -> c
curry f x y = f (x, y)

uncurry :: (a -> b -> c) -> ((a, b) -> c)
uncurry f p = f (fst p) (snd p)

-- Operations on sums and products

(<*>) :: (a -> b) -> (c -> d) -> (a,c) -> (b,d)
(f <*> g) (x,x') = (f x, g x')

(f \ / g) :: (a -> b) -> (a -> c) -> a -> (b,c)
(f \ / g) x = (f x, g x)

(<|>) :: (a -> b) -> (c -> d) -> Either a c -> Either b d
(f <|> g) (Left x) = Left (f x)
(f <|> g) (Right x') = Right (g x')

(f \ / g) :: (a -> c) -> (b -> c) -> Either a b -> c
(f \ / g) (Left x) = f x
(f \ / g) (Right y) = g y

```

Fig. 2. Folklore notation used in point-free definitions.

are effectively functional objects, i.e., they return a “modified (copy of) self”, when mutations are to be modeled.

3.2 Object construction by unfolding

Given a coalgebra $x \rightarrow f x$ for some fixed f and x , we can *unfold* a value of type x to the type of the fixed point of f . The corresponding well-known recursion scheme of *anamorphisms* [26] is instantiated for IExpr as follows:

```

unfoldIExpr :: IExprCoalg x -> x -> IExpr
unfoldIExpr c = InIExpr . fmapIExprF (unfoldIExpr c) . c
where
  fmapIExprF :: (x -> y) -> IExprF x -> IExprF y
  fmapIExprF f = id <*> (.) f

```

That is, IExprF x is injected into IExpr by recursively unfolding all occurrences of x by means of the functorial map operation, fmapIExprF. The type parameter x occurs in the positions where “a modified (copy of) self” is returned. In OO terms, applications of unfoldIExpr are to be viewed as *constructor* methods:

```
newNum :: Int → IExpr
newNum = unfoldIExpr numCoalg
```

```
newAdd :: (IExpr, IExpr) → IExpr
newAdd = unfoldIExpr addCoalg
```

This completes the transcription of the Java code to the coalgebraic setup.

3.3 Converting state trees to object trees

In establishing a semantical correspondence between functional and OO programs, we can exploit the following property: *the functional style is based on recursion into terms whose structure coincides with the states of the objects*. Hence, let us try to convert such trees of states of objects (state trees) into trees of objects (object trees). We will then need to compare the semantics of the resulting objects with the semantics of the functional program.

In the introduction, we defined the data type `Expr` as an algebraic data type; for the sake of a more basic notation, we define it here as the fixed point of a “sum-of-products” functor `ExprF` equipped with convenience injections `num` and `add` — reminiscent of the algebraic data-type constructors of the richer notation. Thus:

```
type ExprF x = Either Int (x,x)
newtype Expr = InExpr { outExpr :: ExprF Expr }
num = InExpr . Left
add = InExpr . Right
```

The objects we construct have internal state either of type `Int` or of type `(IExpr,IExpr)`. The corresponding sum, `Either Int (IExpr,IExpr)`, coincides with `ExprF IExpr`, i.e., the mere state trees resemble the term structure in the functional program. We have defined coalgebras `numCoalg` and `addCoalg` for each type of state. We can also define a coalgebra for the union type `ExprF IExpr`:

```
eitherCoalg :: IExprCoalg (ExprF IExpr)
eitherCoalg = ((id <*> (.) Left) ∨ (id <*> (.) Right)) .
              (numCoalg <|> addCoalg)
```

The coalgebra `eitherCoalg` may be viewed as an implementation of an object type that physically uses a sum of the earlier state types. Alternatively, we may view the coalgebra as an object factory (in the sense of the *abstract factory* design pattern [14]). That is, we may use it as a means to construct objects of either type.

```
newEither :: ExprF IExpr → IExpr
newEither = unfoldIExpr eitherCoalg
```

It is important to understand the meaning of `ExprF IExpr`: the type describes states of objects, where the state representation is only exposed at the top-level, but all deeper objects are already opaque and properly annotated with behavior. While `newEither` facilitates one level of object construction, we ultimately seek the recursive closure of this concept. That is, we seek to convert a pure state tree to a proper object tree. It turns out that the fold operation for `Expr` immediately serves this purpose.

In general, the fold operation for a given sums-of-products functor is parametrized by an *algebra* that associates each addend of the functor's sum with a function that combines recursively processed components and other components. (The notion of algebra, essentially, only means here that the type is of the shape $f\ x \rightarrow x$, while f is a functor.) For instance, the algebra type for expressions is the following:

```
type ExprAlg x = ExprF x → x
```

Given an algebra $f\ x \rightarrow x$ for some fixed f and x , we can *fold* a value of the type of the fixed point of f to a value of type x . The corresponding well-known recursion scheme of *catamorphisms* is instantiated for Expr as follows:

```
foldExpr :: ExprAlg x → Expr → x
foldExpr a = a . fmapExprF (foldExpr a) . outExpr
where
  fmapExprF :: (x → y) → ExprF x → ExprF y
  fmapExprF f = id <|> (f <*> f)
```

We should simplify the earlier type for newEither as follows:

```
newEither :: ExprAlg IExpr
```

Hence, we can fold over state trees to obtain object trees.

```
-- Fold the unfold
fu :: Expr → IExpr
fu = foldExpr newEither
```

3.4 Implementing interfaces by folds over state trees

It remains to compare the semantics of the constructed objects with the semantics of the corresponding functional program. To this end, we also model the construction of objects whose object type corresponds to an abstract data type (ADT) that exports the functions of the functional program as its operations.

The initial definitions of the functions `eval` and `modn` used general recursion. Our development relies on the fact that the functions are catamorphisms (subject to further restrictions). Here are the new definitions; subject to certain preconditions, programs that use general recursion can be automatically converted to programs that use the catamorphic scheme [22,15]:

```
evalAlg :: ExprAlg Int
evalAlg = numEval \ / addEval
```

```
eval :: Expr → Int
eval = foldExpr evalAlg
```

```
modnAlg :: ExprAlg (Int → Expr)
modnAlg = ((.) num . numModn) \ / ((.) add . addModn)
```

```
modn :: Expr → Int → Expr
modn = foldExpr modnAlg
```


Just like objects combine behavior for all operations in their interfaces, we may want to tuple the folds, such that all recursions are performed simultaneously. That is, the result type of the paired fold is the product of the result types of the separated folds. Again such pairing (tupling) is a well-understood technique [13,26,5,16] that can also be used in an automated transformation. Thus:

```
bothAlg :: ExprAlg (IExprF Expr)
bothAlg = (evalAlg <*> modnAlg) . ((id <|> (fst <*> fst)) /\
    (id <|> (snd <*> snd)))
```

```
both :: IExprCoalg Expr
both = foldExpr bothAlg
```

That is, `both` does both, `eval` and `modn`. Now we can construct objects whose behavior is immediately defined in terms of `both` (hence, essentially, in terms of the original functions `eval` and `modn`). To this end, it is sufficient to realize that `both` readily fits as a coalgebra, as evident from its type:

$$\text{Expr} \rightarrow (\text{Int}, \text{Int} \rightarrow \text{Expr}) \equiv \text{Expr} \rightarrow \text{IExprF Expr} \equiv \text{IExprCoalg Expr}$$

Thus, we can construct objects (ADTs, in fact) as follows:

```
-- Unfold the fold
uf :: Expr → IExpr
uf = unfoldIExpr both
```

That is, we have encapsulated the functional folds with an argument term such that the resulting interface admits the applications of the functional folds to the term.

3.5 The expression lemma

Let us assume that we were able to prove the following identity:

$$\text{foldExpr (unfoldIExpr eitherCoalg)} = \text{unfoldIExpr (foldExpr bothAlg)}$$

We refer to the generic form of this identity as the expression lemma; c.f. § 4.4. Roughly, the claim means that *objects that were constructed from plain state trees, level by level, behave the same as shallow objects that act as abstract data types with the functional folds as operations*. Hence, this would define a proper correspondence between anamorphically (and coalgebraically) phrased OO programs and catamorphically phrased functional programs. § 4 formalizes this intuition and proves its validity.

The formal development will exploit a number of basic categorical tools, but a key insight is that the defining coalgebra of the OO program (i.e., `eitherCoalg`) and the defining algebra of the functional program (i.e., `bothAlg`) essentially involve the same function (in fact, a *natural transformation*). To see this, consider the expanded types of the (co)algebras in question:

```
eitherCoalg :: ExprF IExpr → IExprF (ExprF IExpr)
bothAlg :: ExprF (IExprF Expr) → IExprF Expr
```

The types differ in the sense that `eitherCoalg` uses the recursive closure `IExpr` in one position where `bothAlg` uses an application of the functor `IExprF` instead, and `bothAlg` uses

the recursive closure `Expr` in another position where `eitherCoalg` uses an application of the functor `ExprF` instead. Assuming a natural transformation `lambda`, both functions can be defined as follows:

```
eitherCoalg = lambda . fmapExprF outIExpr
bothAlg     = fmapIExprF InExpr . lambda
lambda :: ExprF (IExprF x) -> IExprF (ExprF x)
lambda = ???
```

Note that naturality of `lambda` is essential here. It turns out that we can define `lambda` in a “disjunctive normal form” over the same ingredients that we also used in the original definitions of `eitherCoalg` and `bothAlg`:

```
lambda = (numEval /\ ((.) Left . numModn)) \\/ ((addEval . (fst <*> fst)) /\
        ((.) Right . addModn . (snd <*> snd)))
```

It is straightforward to see that `eitherCoalg` and `bothAlg` as redefined above equate to the original definitions of `eitherCoalg` and `bothAlg` based on just trivial laws for sums and products. That is, the following laws are sufficient to rewrite the original definitions of `eitherCoalg` and `bothAlg` to `eitherCoalg'` and `bothAlg'`:

```
(a / b) . (c <|> d) = (a . c) / (b . d)
(a <*> b) . (c / d) = (a . c) / (b . d)
(a / b) / (c / d) = (a / c) / (b / d)
id . a = a
a . id = a
```

We have thus factored out a common core, a distributive law, `lambda`, from which both programs can be canonically defined.

4 The basic categorical model

The intuitions of the previous section will now be formalized categorically. The formal development will be annotated by examples that refer back to the Haskell illustrations. The used categorical tools are established in the field of functional programming theory. The contribution of the section lies in leveraging these known tools for the expression lemma.

4.1 Interface functors

Definition 1. An interface functor (or simply interface) is a polynomial endofunctor on a category³ \mathbf{C} of the form

$$O \times M \text{ with } O = \prod_{i \in I} A_i^{B_i} \text{ and } M = \prod_{j \in J} (C_j \times \text{Id})^{D_j}$$

where \prod denotes iterated product, all A s, B s, C s and D s are constant functors, Id is the identity functor and all products and exponents are lifted to functors. I and J are finite sets.

³ For simplicity, in this paper, we assume a category \mathbf{C} with enough structure to support our constructions. The category **SET** is always a safe choice.

Note that here and in the rest of the text we use the exponential notation for the function space as usual. Informally, O collects all “methods” that do not use the type of “self” in their results, as it is the case for observers; M collects all “methods” that return a “mutated (copy of) self” (c.f. the use of “ld”), and possibly additional data (c.f. the C s).

Example 1. !ExprF is an interface functor.

$$\text{!ExprF} = \text{Int} \times \text{Id}^{\text{Int}} \cong \text{Int}^1 \times (1 \times \text{Id})^{\text{Int}}$$

Here, Int denotes the functor that is constantly Int , i.e. $\text{Int } X = \text{Int}$ and $\text{Int } f = \text{id}$. Likewise for 1 .

4.2 F -(co)algebras and their morphisms

Definition 2. Let F be an endofunctor on a category \mathbf{C} , A, B objects in \mathbf{C} .

- An F -**algebra** is an arrow $F A \longrightarrow A$. Here A is called the **carrier** of the algebra.
- For F -algebras $\varphi : F A \longrightarrow A$ and $\psi : F B \longrightarrow B$, an F -**algebra morphism** from φ to ψ is an arrow $f : A \longrightarrow B$ of \mathbf{C} such that the following holds:

$$f \circ \varphi = \psi \circ F f \quad (1)$$

$$\begin{array}{ccc} F A & \xrightarrow{F f} & F B \\ \varphi \downarrow & & \downarrow \psi \\ A & \xrightarrow{f} & B \end{array}$$

- F -algebras and F -algebra morphisms form a category denoted \mathbf{C}^F . The initial object in this category, if it exists, is the **initial F -algebra**. Explicitly, it is an F -algebra, $\text{in}_F : F \mu F \longrightarrow \mu F$, such that for any other F -algebra, $\varphi : F A \longrightarrow A$, there exists a unique F -algebra morphism $(\varphi)_F$ from in_F to φ . Equivalently:

$$h = (\varphi)_F \Leftrightarrow h \circ \text{in}_F = \varphi \circ F h \quad (2)$$

$$\begin{array}{ccc} F \mu F & \xrightarrow{F (\varphi)_F} & F A \\ \text{in}_F \downarrow & & \downarrow \varphi \\ \mu F & \xrightarrow{(\varphi)_F} & A \end{array}$$

- The duals of the above notions are F -**coalgebra**, F -**coalgebra morphism** and the **terminal F -coalgebra**. Explicitly, an F -coalgebra is an arrow $\varphi : A \longrightarrow F A$ in \mathbf{C} . The category of F -coalgebras is denoted \mathbf{C}_F . The terminal F -coalgebra is denoted $\text{out}_F : \nu F \longrightarrow F \nu F$, and the unique terminal F -coalgebra morphism from φ is denoted $(\varphi)_F : A \longrightarrow \nu F$. These satisfy the following duals of (1) and (2).

$$\psi \circ g = F g \circ \varphi \quad (3)$$

$$h = (\varphi)_F \Leftrightarrow \text{out}_F \circ h = F h \circ \varphi \quad (4)$$

$$\begin{array}{ccc}
A & \xrightarrow{g} & B \\
\varphi \downarrow & & \downarrow \psi \\
FA & \xrightarrow{Fg} & FB
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\llbracket \varphi \rrbracket_F} & \nu F \\
\varphi \downarrow & & \downarrow \text{out}_F \\
FA & \xrightarrow{F \llbracket \varphi \rrbracket_F} & F \nu F
\end{array}$$

Example 2. (Again, we relate to the Haskell declarations of § 3.) ExprF is an endofunctor; μ_{ExprF} corresponds to type Expr ; evalAlg and modnAlg are ExprF -algebras. The combinator $\llbracket _ \rrbracket_{\text{ExprF}}$ corresponds to foldExpr . Likewise, IExprF is an endofunctor; ν_{IExprF} corresponds to IExpr ; numCoalg and addCoalg are IExprF -coalgebras. The combinator $\llbracket _ \rrbracket_{\text{IExprF}}$ corresponds to unfoldIExpr .

4.3 Simple distributive laws

Our approach to proving the correspondence between OO and functional programs critically relies on distributive laws as a means to relate algebras and coalgebras. In the present section, we only introduce the simplest form of distributive laws.

Definition 3. A *distributive law of a functor F over a functor B* is a natural transformation $FB \rightarrow BF$.

Such simple distribute laws of a functor over another functor are sufficient for the motivating example of § 3. Other couples of functional and OO programs call for distributive laws of functors with additional structure (such as monads and comonads), and § 6 will advance the formalization accordingly.

Example 3. Trivial examples of distributive laws are algebras and coalgebras. That is, any coalgebra $X \rightarrow BX$ is a distributive law where F in Def. 3 is fixed to be a constant functor. Dually, any algebra $FX \rightarrow X$ is a distributive law where B in Def. 3 is fixed to be a constant functor. This fact is convenient in composing algebras and coalgebras (and distributive laws), as we will see shortly.

Example 4. Here are examples of nontrivial distributive laws:⁴

$$\begin{aligned}
\text{addModn} &: \text{Id}^2 \text{Id}^{\text{Int}} \rightarrow \text{Id}^{\text{Int}} \text{Id}^2 \\
\text{lambda} &: \text{ExprF IExprF} \rightarrow \text{IExprF ExprF}
\end{aligned}$$

Distributive laws can be combined in various ways, e.g., by \oplus and \otimes defined as follows:

Definition 4. Let $\lambda_i : F_i B \rightarrow B F_i$, $i \in \{1, 2\}$ be distributive laws. Then we define a distributive law:

$$\begin{aligned}
\lambda_1 \oplus \lambda_2 &: (F_1 + F_2) B \rightarrow B (F_1 + F_2) \\
\lambda_1 \oplus \lambda_2 &\equiv (B \iota_1 \nabla B \iota_2) \circ (\lambda_1 + \lambda_2)
\end{aligned}$$

⁴ Note that we use just juxtaposition for functor composition. Confusion with application is not an issue because application can be always considered as composition with a constant functor. Also note that $F^2 \cong F \times F$ in a bicartesian closed category.

Here, $f \nabla g$ is the cotuple of f and g with injections ι_1 and ι_2 , that is the unique arrow such that $(f \nabla g) \circ \iota_1 = f$ and $(f \nabla g) \circ \iota_2 = g$.

Definition 5. Let $\lambda_i : FB_i \rightarrow B_iF$, $i \in \{1, 2\}$ be distributive laws. Then we define a distributive law:

$$\begin{aligned} \lambda_1 \otimes \lambda_2 &: F(B_1 \times B_2) \rightarrow (B_1 \times B_2)F \\ \lambda_1 \otimes \lambda_2 &\equiv (\lambda_1 \times \lambda_2) \circ (F\pi_1 \Delta F\pi_2) \end{aligned}$$

Here, $f \Delta g$ is the tuple of f and g with projections π_1 and π_2 , that is the unique arrow such that $\pi_1 \circ (f \Delta g) = f$ and $\pi_2 \circ (f \Delta g) = g$.

We assume the usual convention that \otimes binds stronger than \oplus .

Example 5. As algebras and coalgebras are distributive laws, \oplus and \otimes readily specialize to combinators on algebras and coalgebras, as in `bothAlg` or `eitherCoalg`. A nontrivial example of a combination of distributive laws is `lambda`:

$$\text{lambda} = \text{numEval} \otimes \text{numModn} \oplus \text{addEval} \otimes \text{addModn} \quad (5)$$

The following lemma states a basic algebraic property of \oplus and \otimes .

Lemma 1. Let $\lambda_{i,j} : F_{i,j}B_{i,j} \rightarrow B_{i,j}F_{i,j}$, $i, j \in \{1, 2\}$ be distributive laws. Then:

$$(\lambda_{1,1} \otimes \lambda_{2,1}) \oplus (\lambda_{1,2} \otimes \lambda_{2,2}) = (\lambda_{1,1} \oplus \lambda_{1,2}) \otimes (\lambda_{2,1} \oplus \lambda_{2,2})$$

Proof. By elementary properties of tuples and cotuples, in particular, by the following law (called the ‘‘abides law’’ in [26]):

$$(f \Delta g) \nabla (h \Delta i) = (f \nabla h) \Delta (g \nabla i)$$

□

Example 6. By the above lemma (compare with (5)):

$$\text{lambda} = (\text{numEval} \oplus \text{addEval}) \otimes (\text{numModn} \oplus \text{addModn}) \quad (6)$$

Examples 5 and 6 illustrate the duality between the functional and OO approaches to *program decomposition*. In functional programming, different cases of the same function, each for a different component of the top-level disjoint union of an algebraic data type, are cotupled by case distinction (c.f. occurrences of \oplus in (6)). In contrast, in OO programming, functions on the same data are tupled into object types (c.f. occurrences of \otimes in (5)).

4.4 The simple expression lemma

We have shown that we may extract a natural transformation from the algebra of a functional fold that can be reused in the coalgebra of an unfold for object construction. It

remains to be shown that the functional fold and the OO unfold are indeed semantically equivalent in such a case.

Given a distributive law $\lambda : FB \rightarrow BF$, one can define an arrow $\mu F \rightarrow \nu B$ by the following derivation:

$$\lambda_{\nu B} \circ F\text{out}_B : F\nu B \rightarrow BF\nu B \quad (7)$$

$$\llbracket \lambda_{\nu B} \circ F\text{out}_B \rrbracket_B : F\nu B \rightarrow \nu B \quad (8)$$

$$\langle \langle \llbracket \lambda_{\nu B} \circ F\text{out}_B \rrbracket_B \rangle \rangle_F : \mu F \rightarrow \nu B \quad (9)$$

An example of (7) is either `Coalg` in §3.5.

Dually, the following also defines an arrow $\mu F \rightarrow \nu B$:

$$\text{Bin}_F \circ \lambda_{\mu F} : FB\mu F \rightarrow B\mu F \quad (10)$$

$$\langle \langle \text{Bin}_F \circ \lambda_{\mu F} \rangle \rangle_F : \mu F \rightarrow B\mu F \quad (11)$$

$$\llbracket \langle \langle \text{Bin}_F \circ \lambda_{\mu F} \rangle \rangle_F \rrbracket_B : \mu F \rightarrow \nu B \quad (12)$$

An example is both `Alg` in §3.5.

The following theorem shows that (12) is equal to (9) and thus, as discussed in §3.5, establishes a formal correspondence between anamorphically phrased OO programs and catamorphically phrased functional programs.

Theorem 1 (“Simple expression lemma”). *Let out_B be the terminal B -coalgebra and in_F be the initial F -algebra. Let $\lambda : FB \rightarrow BF$. Then*

$$\langle \langle \llbracket \lambda_{\nu B} \circ F\text{out}_B \rrbracket_B \rangle \rangle_F = \llbracket \langle \langle \text{Bin}_F \circ \lambda_{\mu F} \rangle \rangle_F \rrbracket_B$$

Proof. We show that the right-hand side, $\llbracket \langle \langle \text{Bin}_F \circ \lambda_{\mu F} \rangle \rangle_F \rrbracket_B$, satisfies the universal property of the left-hand side; c.f. (2):

$$\langle \langle \llbracket \lambda_{\nu B} \circ F\text{out}_B \rrbracket_B \rangle \rangle_F \circ F \llbracket \langle \langle \text{Bin}_F \circ \lambda_{\mu F} \rangle \rangle_F \rrbracket_B = \llbracket \langle \langle \text{Bin}_F \circ \lambda_{\mu F} \rangle \rangle_F \rrbracket_B \circ \text{in}_F \quad (13)$$

The calculation is straightforward by a two-fold application of the following rule, called “AnaFusion” in [26]:

$$\langle \langle \varphi \rangle \rangle_B \circ f = \langle \langle \psi \rangle \rangle_B \iff \varphi \circ f = Bf \circ \psi \quad (14)$$

The premise of the rule is precisely the statement that f is a B -coalgebra morphism to φ from ψ . The proof is immediate by compositionality of coalgebra morphisms and uniqueness of the universal arrow. It’s worth mentioning at this point that there is also the following dual rule called “CataFusion” for catamorphisms. We won’t need it in this proof.

$$f \circ \langle \langle \varphi \rangle \rangle_F = \langle \langle \psi \rangle \rangle_F \iff f \circ \varphi = \psi \circ Ff \quad (15)$$

Using this rule we proceed as follows:

$$\begin{aligned} & \llbracket \lambda \circ F\text{out}_B \rrbracket_B \circ F \llbracket \langle \langle \text{Bin}_F \circ \lambda \rangle \rangle_F \rrbracket_B \\ = & \quad \{ \text{By (14) and the following:} \end{aligned}$$

$$\begin{aligned}
& \lambda \circ F \text{out}_B \circ F \llbracket (\text{Bin}_F \circ \lambda)_F \rrbracket_B \\
= & \{ \text{functor composition} \} \\
& \lambda \circ F(\text{out}_B \circ \llbracket (\text{Bin}_F \circ \lambda)_F \rrbracket_B) \\
= & \{ \llbracket \dots \rrbracket_B \text{ is a coalgebra morphism} \} \\
& \lambda \circ F(B \llbracket (\text{Bin}_F \circ \lambda)_F \rrbracket_B \circ (\text{Bin}_F \circ \lambda)_F) \\
= & \{ \lambda \text{ is natural} \} \\
& (BF \llbracket (\text{Bin}_F \circ \lambda)_F \rrbracket_B) \circ \lambda \circ F(\text{Bin}_F \circ \lambda)_F \} \\
& \llbracket \lambda \circ F(\text{Bin}_F \circ \lambda)_F \rrbracket_B \\
= & \{ \text{By (14) and the following fact:} \\
& (\text{Bin}_F \circ \lambda)_F \circ \text{in}_F \\
= & \{ (\dots)_F \text{ is a } F\text{-algebra morphism} \} \\
& \text{Bin}_F \circ \lambda \circ F(\text{Bin}_F \circ \lambda)_F \} \\
& \llbracket (\text{Bin}_F \circ \lambda)_F \rrbracket_B \circ \text{in}_F \quad \square
\end{aligned}$$

5 Classes of dualizable folds

The formal development, so far, establishes that a functional fold $(\varphi)_F : \mu F \longrightarrow B\mu F$ is dualizable to OO, if there is a distributive law (i.e., so far, a natural transformation) $\lambda : FB \longrightarrow BF$ such that $\varphi = \text{Bin}_F \circ \lambda$. §6 will continue the formal development to admit more general distributive laws — by adding extra structure to the functors, based on the notions of monads and comonads.

The present section illustrates important classes of folds that are covered by the formal development of this paper (including the elaboration to be expected from §6). For each class of folds, we introduce a variation on the plain fold operation so that the characteristics of the class are better captured. The first argument of a varied fold operation is not a fold algebra formally; it is rather used for composing a proper fold algebra, which is to be passed to the plain fold operation.

5.1 Void folds

Consider again the Java rendering of the modn function:⁵

```

public abstract void Expr.modn(int v); //Modify literals modulo v
public void Num.modn(int v) { this.value = this.value % v; }
public void Add.modn(int v) { left.modn(v); right.modn(v); }

```

⁵ We use a concise OO notation such that the hosting class of an instance method is simply shown as a qualifier of the method name when giving its signature and implementation. This notation can be interpreted to facilitate modular class extensions, as provided by a number of OO language designs [20,41,42].

That is, the `modn` method needs to mutate the value fields of all `Num` objects, but no other changes are needed. Hence, the imperative OO style suggests to defer to a method without a proper result type, i.e., a void method. In the reading of functional objects, a void method has a result type that is equal to the type parameter of the interface functor. There is a restricted fold operation that captures the idea of voidity in a functional setup:

```
type VExprArg x =
  (Int → x → Int,
   (x → Expr, x → Expr) → x → (Expr, Expr))
vFoldExpr :: VExprArg x → Expr → x → Expr
vFoldExpr a = foldExpr (((.) num . fst a) \ / ((.) add . snd a))
```

The void fold operation takes a product — one type-preserving function for each data variant. The type parameter `x` enables void folds with extra arguments. (The plain fold operation possesses the argument capability without any precautions because of its polymorphic result type, which is not the case for the void variation.) To aid understanding, here is a simplified variation without such argument capability:

```
type VExprArg' = (Int → Int, (Expr, Expr) → (Expr, Expr))
vFoldExpr' :: VExprArg' → Expr → Expr
vFoldExpr' a = foldExpr ((num . fst a) \ / (add . snd a))
```

For instance, the `modn` function can be phrased as a void fold with an argument of type `Int`:

```
modn :: Expr → Int → Expr
modn = vFoldExpr (numModn, addModn)
```

Rewriting a general fold to a void fold requires nothing more than factoring the general fold algebra so that it follows the one that is composed in the `vFoldExpr` function above. That is, for each constructor, its case preserves the constructor. A void fold must also be sufficiently natural in order to be dualizable; c.f. the next subsection.

A note on mutable objects Void folds are still functional folds; they just make explicit that the outermost constructor is preserved; they are dualized as methods of functional object types that return a modified copy. In an imperative OO language, void methods are intrinsically impure; they rely on in-place mutations of objects, and our formal development cannot be applied, as is. In particular, given a functional program p that *uses a void fold f* over (immutable) algebraic data, we cannot translate p to a semantically equivalent imperative OO program that *uses the corresponding void method m* over the associated recursive object-type structure (of mutable objects), without accounting for the possible side effects that are implied by any use of m .

5.2 Natural folds

Consider again the type of the fold algebra for the `modn` function as introduced in § 3.4:

```
modnAlg :: ExprAlg (Int → Expr)
≡ modnAlg :: ExprF (Int → Expr) → (Int → Expr)
```


The type admits observation of the precise structure of intermediate results; c.f. the occurrences of `Expr`. This capability would correspond to unlimited introspection in OO programming (including “instance of” checks and casts). The formal development requires that the algebra must be amenable to factoring as follows:

```
modnAlg = fmaplExprF lInExpr . lambda
  where
    lambda :: ExprF (Int → x) → Int → ExprF x
    lambda = ...
```

That is, the algebra is only allowed to observe one layer of functorial structure. In fact, it is easy to see that the actual definition of `modnAlg` suffices with this restriction. We may want to express that a fold is readily in a “natural form”. To this end, we may use a varied fold operation whose argument type is accordingly parametric:⁶

```
type NExprArg x = forall y. ExprF (x → y) → x → ExprF y
nExpr :: NExprArg x → Expr → x → Expr
nExpr a = foldExpr ((.) lInExpr . a)
```

It is clear that the type parameter `x` is used at the type `Expr`, but universal quantification rules out any exploitation of this fact, thereby enabling the factoring that is required by the formal development. For completeness’ sake, we also provide a voidity-enforcing variation; it removes the liberty of replacing the outermost constructor:

```
type VnExprArg x = forall y. (Int → x → Int, (x → y, x → y) → x → (y, y))
vnExpr :: VnExprArg x → Expr → x → Expr
vnExpr a = foldExpr ((.) lInExpr . (((.) Left . fst a) \ / ((.) Right . snd a)))
```

Here is also a simplified version for type-preserving folds without arguments:

```
type VnExprArg' = forall x. (Int → Int, (x, x) → (x, x))
vnExpr' :: VnExprArg' → Expr → Expr
vnExpr' a = foldExpr (lInExpr . (fst a <|> snd a))
```

The `modn` function is a void, natural fold:

```
modn :: Expr → Int → Expr
modn = vnExpr (numModn, addModn)
```

Naturality disallows us to perform any operation on intermediate results. However, we are allowed to exploit the fact that all intermediate results are of the same type, as illustrated by the following function which swaps the operands of each addition.

```
-- Defined in terms of general recursion
swap :: Expr → Expr
swap (Num i) = Num i
swap (Add l r) = Add (swap r) (swap l)

-- Defined as a void, natural fold
swap :: Expr → Expr
swap = vnExpr' (id, snd / \ fst)
```

⁶ We use a popular Haskell 98 extension for rank-2 polymorphism; c.f. `forall`.

The OO counterpart looks as follows:

```

public abstract void Expr.swap();
public void Num.swap() {}
public void Add.swap() {
    left.swap(); right.swap();
    Expr x = left; left = right; right = x;
}

```

The distributive law of the formal development, i.e., $\lambda : FB \rightarrow BF$, is more general than the kind of natural F -folds that we illustrated above. That is, λ is not limited to type-preserving F -folds, but it uses the extra functor B to define the result type of the F -fold in terms of F . This extra functor is sufficient to cover “constant folds” (such as `eval`), “paramorphic folds” [25] (i.e., folds that also observe the unprocessed, immediate components) and “tupled folds” (i.e., folds that were composed from separated folds by means of tupling). The source distribution of the paper illustrates all these aspects.

5.3 Free monadic folds

The type of the natural folds considered so far implies that *intermediate results are to be combined in exactly one layer of functorial structure*, c.f. the use of `ExprF` in the result-type position of `NExprArg`. The formal development will waive this restriction in § 6. Let us motivate the corresponding generalization.

The generalized scheme of composing intermediate results is to arbitrarily nest constructor applications, including the base case of returning one recursive result, as is. Consider the following function that returns the leftmost expression; its `Add` case does not replace the outermost constructor; instead, the outermost constructor is dropped:

```

leftmost :: Expr → Expr
leftmost (Num i) = Num i
leftmost (Add l r) = leftmost l

```

The function cannot be phrased as a natural fold. We can use a plain fold, though:

```

leftmost = foldExpr (num \_/ fst)

```

The following OO counterpart is suggestive:

```

public abstract Expr Expr.leftmost();
public Expr Num.leftmost() { return this; }
public Expr Add.leftmost() { return left.leftmost(); }

```

Consider the following function for “exponential cloning”; it combines intermediate results in a *nest* of constructor applications (while the `leftmost` function dropped off a constructor):

```

explode :: Expr → Expr
explode x@(Num i) = Add x x
explode (Add l r) = clone (Add (explode l) (explode r))
    where clone x = Add x x

```

Again, the function cannot be phrased as a natural fold. We can use a plain fold:

```

explode :: Expr → Expr
explode = fFoldExpr ((clone . freeNum) \\/ (clone . freeAdd . (var <*> var)))
  where clone = freeAdd . (id /\ id)

```

The following OO counterpart uses the “functional” constructors for the object types.

```

public abstract Expr Expr.explode();
public Expr Num.explode() { return new Add(new Num(value),new Num(value)); }
public Expr Add.explode() {
  return new Add(
    new Add(left.explode(), right . explode()),
    new Add(left.explode(), right . explode ());
}

```

We need a generalized form of natural F -folds where the result of each case may be either of type x , or of type $F x$, or of any type $F^n x$ for $n \geq 2$. The corresponding union of types is modeled by the free type of F , i.e., the type of free terms (variable terms) over F . This type is also known as the free monad. For instance, the free type of expressions is defined as follows:

```

newtype FreeExpr x = InFreeExpr { outFreeExpr :: Either x (ExprF (FreeExpr x)) }

```

We also assume the following convenience injections:

```

var = InFreeExpr . Left
term = InFreeExpr . Right
freeNum = term . Left
freeAdd = term . Right

```

Natural folds are generalized as follows:

```

type FExprArg x = forall x. ExprF x → FreeExpr x
fFoldExpr :: FExprArg x → Expr → Expr
fFoldExpr a = foldExpr (collapse . a)
  where
    collapse :: FreeExpr Expr → Expr
    collapse = (id \\/ (leaf \\/ (fork . (collapse <*> collapse)))) . outFreeExpr

```

(For simplicity, we only consider folds without extra arguments here.) That is, we compose together a fold algebra that first constructs free-type terms from intermediate results, and then collapses the free-type layers by conversion to the recursive closure of the functor at hand. Term construction over the free type is to be dualized to object construction. The two motivating examples can be expressed as “free” natural folds:

```

leftmost = fFoldExpr (freeNum \\/ (var . fst))
explode = fFoldExpr ((clone . freeNum) \\/ (clone . freeAdd . (var <*> var)))
  where clone = freeAdd . (id /\ id)

```

A note on sharing and cycles The careful reader may observe that the current discussion brings up the tension between tree shape vs. DAG shape (due to sharing, say aliasing), or general graph shape (including cycles) for object structures. Clearly, our coalgebraic OO model assumes tree shape, but it is clear that an imperative OO language may

intentionally or accidentally use sharing and cycles. For instance, the illustrating Java method `Num.leftmost` may lead to sharing, if the result of the method invocation is stored in another object structure. Combined with object mutation, sharing may carry semantic meaning. The illustrating Java methods `Num.explode` and `Add.explode` readily perform deep cloning, so as to counter the possibility of sharing. However, a native OO programmer may prefer a non-cloning solution for reasons of memory footprint or intentional sharing.

5.4 Cofree comonadic folds

The type of the natural folds considered so far implies that *the algebra has access to the results of applying the recursive function to immediate components exactly once*. The formal development will waive this restriction in § 6. Let us motivate the corresponding generalization.

In general, we may want to apply the recursive function any number of times to each immediate component. We may think of such expressiveness as an iteration capability. Here is a very simple example of a function that increments twice on the left, and once on the right:

```
leftist  :: Expr → Expr
leftist (Num i) = Num (i+1)
leftist (Add l r) = Add (leftist (leftist l)) (leftist r)
```

Such iteration is also quite reasonable in OO programming:

```
public abstract void Expr.leftist ();
public void Num.leftist () { value++; }
public void Add.leftist () { left.leftist (); left.leftist (); right.leftist (); }
```

The simple example only involves a single recursive function, and hence, arbitrary repetitions of that function can be modeled as a *stream* (i.e., a coinductive list). A designated, streaming-enabled fold operation, `sFoldExpr`, deploys a stream type as the result type. The argument of such a fold operation can select among different numbers of repetitions in the following style:

```
leftist  :: Expr → Expr
leftist = sFoldExpr ((+1) <|> ((head . tail . tail) <*> (head . tail)))
```

Here, `head` maps to “0 applications”, `head . tail` maps to “1 application”, `head . tail . tail` maps to “2 applications”. The streaming-enabled fold operation composes together a fold algebra that produces a stream of results at each level of folding. To this end, we need the coinductive unfold operation for streams:

```
type StreamCoalg x y = y →(x,y)
unfoldStream :: StreamCoalg x y →y → [x]
unfoldStream c = uncurry (:). (id <*> unfoldStream c) . c
```

The streaming-enabled fold operation is defined as follows:

```
type SExprArg = forall x. ExprF [x] → ExprF x
sFoldExpr :: SExprArg →Expr →Expr
```

```

sFoldExpr a = head . tail . foldExpr a'
  where
    a' :: ExprF [Expr] → [Expr]
    a' = unfoldStream (hd /\ tl)
    hd = InExpr . (id <|> (head <*> head))
    tl = a . (id <|> (iterate tail <*> iterate tail ))

```

The argument type of the operation, `SExprArg`, can be interpreted as follows: given a stream of repetitions for each immediate component, construct a term with a selected number of repetitions for each immediate component position. In fact, the selection of the favored number of repetitions is done by cutting of only the disfavored prefix of the stream (as opposed to actual selection, which would also cut off the postfix). Thereby, iteration is enabled; each step of iteration further progresses on the given stream.

If we look closely, we see that the argument `a :: SExprArg` is not provided with a *flat* stream of repetitions but rather *a stream of streams* of remaining repetitions; c.f. the use of the standard function, `iterate`, for coinductive iteration, which we recall here for the reader's convenience:

```

iterate :: (a → a) → a → [a]
iterate f x = x : iterate f (f x)

```

The type `SExprArg` protects the nesting status of the stream by universal quantification, thereby avoiding that the stream of remaining repetitions is manipulated in an undue manner such as by reshuffling. For comparison, an unprotective type would be the following:

```

forall x. ExprF [[x]] → ExprF [x]

```

When multiple (mutually recursive) functions are considered, then we must turn from a stream of repetitions to infinite trees of repeated applications; each branch corresponds to the choice of a particular mutation or observation. This tree structure would be modeled by a functor, reminiscent of an interface functor, and the stream type is generalized to the cofree comonad over a functor.

Let us consider an example that involves multiple functions. The following function increments “many times” on the left, and never on the right. The number of repetitions on the left is bounded by the `eval` observation on immediate results. Hence, we face two functions, where the recursive results of one function depend on the other function.

```

many :: Expr → Expr
many (Num i) = Num (i+1)
many (Add l r) = Add (l' star l) r
  where
    l' star l = let l' = many l in
                if eval l' < eval r then l' star l' else l'

```

In Java:

```

public abstract void Expr.many();
public void Num.many() { value++; }
public void Add.many() {
    for ( ; ; ) {

```

```

    left .many();
    if (!(left .eval() < right .eval ())) break;
  }
}

```

A designated fold operation would allow its argument to select any number of repetitions of many and the value of the eval observation for any selected number. Hence, the infinitary tree structure is still degenerated because we think of a stream of pairs. Once we face multiple mutations, we need a proper tree structure where multiple branches are infinitary.

6 The categorical model continued

We will now extend the theory of § 4 in order to cater for the examples of § 5. In particular our notion of a dualizable program, which used to be a simple natural transformation of type $FB \rightarrow BF$, will be extended to more elaborate distributive laws between a monad and a comonad [2,36]. This extra structure provides the power needed to cover functional folds that iterate term construction or recursive function application. We show that all concepts from § 4 lift appropriately to (co)monads. This includes (co)algebras, morphisms, and most notably, distributive laws. We also provide means to construct the generalized distributive laws from more manageable natural transformations, which are the key to programming with the theory and justify the examples of § 5. We eventually generalize the final theorem of § 4 about the semantical correspondence between functions and objects. The used categorical tools are relatively straightforward and well known; [1] is an excellent reference for this section. We only claim originality in their adoption to the semantical correspondence problem of the expression lemma.

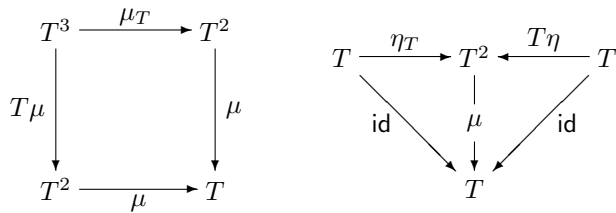
6.1 ((Co)free) (co)monads

We begin with a reminder of the established definitions of (co)monads.

Definition 6. Let \mathbf{C} be a category. A **monad** on \mathbf{C} is a triple $\langle T, \mu, \eta \rangle$, where T is an endofunctor on \mathbf{C} , $\eta : \text{Id}_{\mathbf{C}} \rightarrow T$ and $\mu : T^2 \rightarrow T$ are natural transformations satisfying the following identities:

$$\mu \circ \mu_T = \mu \circ T\mu \tag{16}$$

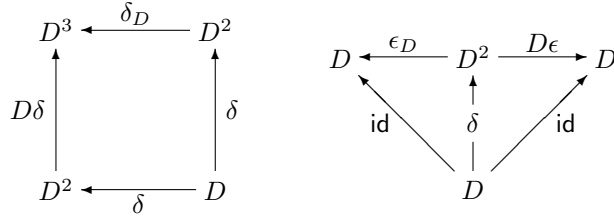
$$\mu \circ \eta_T = \text{id} = \mu \circ T\eta \tag{17}$$



A **comonad** is defined dually as a triple $\langle D, \delta, \epsilon \rangle$ where $\delta : D \longrightarrow D^2$ and $\epsilon : D \longrightarrow \text{Id}_{\mathbf{C}}$ satisfy the duals of (16) and (17):

$$\delta_D \circ \delta = D\delta \circ \delta \quad (18)$$

$$\epsilon_D \circ \delta = \text{id} = D\epsilon \circ \delta \quad (19)$$



The more classical use of monads in functional programming concerns effects, non-determinism and CPS [27,39]. Our development instead focuses on iteration of (i) term construction in the composition of intermediate results, and (ii) recursive function application on intermediate components. These two aspects are modeled by (co)free (co)monads. Here we note that term construction is dualized to OO as object construction (and, more obviously, function application is dualized to OO as method invocation).

Definition 7. Let F be an endofunctor on \mathbf{C} . Let F_X^\dagger be the functor $F_X^\dagger = X + F\text{Id}$. The **free monad** of the functor F is a functor T_F that is defined as follows:

$$\begin{aligned} T_F X &= \mu F_X^\dagger \\ T_F f &= (\text{in}_{F_Y^\dagger} \circ (f + \text{id})) \downarrow_{F_X^\dagger}, \text{ for } f : X \longrightarrow Y \end{aligned}$$

We make the following definitions:

$$\begin{aligned} \eta_X &= \text{in}_{F_X^\dagger} \circ \iota_1 : X \longrightarrow T_F X \\ \tau_X &= \text{in}_{F_X^\dagger} \circ \iota_2 : FT_F X \longrightarrow T_F X \\ \mu_X &= (\text{id}_{T_F X} \nabla \tau_X) \downarrow_{F_X^\dagger} \end{aligned}$$

Now, η and μ are natural transformations, $\langle T_F, \mu, \eta \rangle$ is a monad.

Example 7. We can think of the type $T_F X$ as of the type of non-ground terms generated by signature F with variables from X ; c.f. the Haskell data type `FreeExpr` in § 5.3. Then, η makes a variable into a term (c.f. `var`); τ constructs a term from a constructor in F applied to terms (c.f. `term`), μ is substitution (c.f. `collapse`, which is actually μ_0 ; the type `collapse : ExprF Expr → Expr` reflects the fact that $T_F 0 \cong \mu F$).

Cofree comonads are defined dually.

Definition 8. Let B be an endofunctor on \mathbf{C} , let B_X^\ddagger be the functor $B_X^\ddagger = X \times \text{Bld}$. The *cofree comonad* of the functor B is a functor D_B that is defined as follows:

$$\begin{aligned} D_B X &= \nu B_X^\ddagger \\ D_B f &= \llbracket (f \times \text{id}) \circ \text{out}_{B_X^\ddagger} \rrbracket_{B_Y^\ddagger} \\ \epsilon_X &= \pi_1 \circ \text{out}_{B_X^\ddagger} : D_B X \longrightarrow X \\ \xi_X &= \pi_2 \circ \text{out}_{B_X^\ddagger} : D_B X \longrightarrow B D_B X \\ \delta_X &= \llbracket \text{id}_{D_B X} \Delta \xi_X \rrbracket_{B_{D_B X}^\ddagger} \end{aligned}$$

Example 8. Let us consider a special case: $D_{\text{id}} X$. We can think of this type as the stream of values of type X ; c.f. the coinductive use of the Haskell's list-type constructor in § 5.4. This use corresponds to the following cofree comonad:

$$\text{Stream } X = \nu Y. X \times Y = D_{\text{id}} X$$

Here ϵ is head, ξ is tail, δ is tails : $\text{stream } X \rightarrow \text{stream}(\text{stream } X)$: the function turning a stream into the stream of its tails, i.e. iterate tail. Also note that $D_B 1 \cong \nu B$.

A note on free and non-free monads and comonads In the present paper, we deal exclusively with free constructions (monads and comonads); free constructions have straightforward interpretations as programs. However part of the development, and Theorem 5 in particular, hold in general, and there are interesting examples of non-free constructions arising for instance as quotients with respect to collections of equations.

6.2 (Co)monadic (co)algebras and their morphisms

The notions of folds (and algebras) and unfolds (and coalgebras) lift to monads and comonads. This status will eventually allows us to introduce distributive laws of monads over comonads. We begin at the level of algebras. An algebra of a monad is an algebra of the underlying functor of the monad, which respects the unit and multiplication. Thus:

Definition 9. Let \mathbf{C} be a category, let $\langle T, \eta, \mu \rangle$ be a monad on \mathbf{C} . An T -algebra is an arrow $\alpha : T X \rightarrow X$ in \mathbf{C} such that the following equations hold:

$$\text{id}_X = \alpha \circ \eta_X \tag{20}$$

$$\alpha \circ \mu_X = \alpha \circ T\alpha \tag{21}$$

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & T X \\ & \searrow \text{id}_X & \downarrow \alpha \\ & & X \end{array} \quad \begin{array}{ccc} T^2 X & \xrightarrow{\mu_X} & T X \\ T\alpha \downarrow & & \downarrow \alpha \\ T X & \xrightarrow{\alpha} & X \end{array}$$

A T -algebra morphism between T -algebras $\alpha : T A \rightarrow A$ and $\beta : T B \rightarrow B$ is an arrow $f : A \rightarrow B$ in \mathbf{C} such that

$$f \circ \alpha = \beta \circ T f$$

$$\begin{array}{ccc}
T A & \xrightarrow{Tf} & T B \\
\alpha \downarrow & & \downarrow \beta \\
A & \xrightarrow{f} & B
\end{array}$$

The category of T -algebras for a monad⁷ T is denoted \mathbf{C}^T .

The notion of the *category of D -coalgebras*, \mathbf{C}_D , for a comonad D is exactly dual. We record the following important folklore fact about the relation of (co)algebras of a functor and (co)algebras of its free (co)monad; it allows us to compare the present (co)monadic theory to the simple theory of §4.

Theorem 2. *Let F and B be endofunctors on \mathbf{C} . Then the category \mathbf{C}^F of F -algebras is isomorphic to the category \mathbf{C}^{T_F} of algebras of the free monad. And dually: the category \mathbf{C}_B of B -coalgebras is isomorphic to \mathbf{C}_{D_B} , the category of coalgebras of the cofree comonad D_B .*

Proof. We show the two constructions: from \mathbf{C}^F to \mathbf{C}^{T_F} and back. To this end, let $\varphi : FX \rightarrow X$ be an F -algebra, then $\varphi^* = (\text{id}_X \nabla \varphi)_{F^\dagger_X}$ is a T_F -algebra. (Read φ with superscript $*$ as “lift φ freely”.) In the other direction, given a T_F -algebra α , the arrow $\alpha_* = \alpha \circ \tau_X \circ F\eta_X$ is an F -algebra. (Read α with subscript $*$ as “unlift α freely”.) We omit the check that the two directions are inverse and that φ^* is indeed a T_F -algebra. The dual claim follows by duality: for $\psi : X \rightarrow BX$, $\psi^\alpha = [(\text{id}_X \Delta \psi)]_{B^\ddagger_X}$; $\beta_\alpha = B\epsilon_X \circ \xi_X \circ \beta$.

Example 9. Consider the functor $\text{Expr}F$. Then $T_{\text{Expr}F} X$ is the type of expressions with (formal) variables from X . For $\text{evalAlg} \equiv \text{id} \nabla (\text{curry } (+)) : \text{Int} + \text{Int}^2 \rightarrow \text{Int}$, $\text{evalAlg}^* : T_{\text{Expr}F} \text{Int} \rightarrow \text{Int}$ recursively evaluates an expression where the variables are integers. For $\text{eval}' : T_{\text{Expr}F} \text{Int} \rightarrow \text{Int}$ that evaluates (“free”) expressions, one can reproduce the function $\text{evalAlg} \equiv \text{eval}'_*$ by applying eval' to trivial expressions, provided that eval' is sufficiently uniform (in the sense of equations (20) and (21)).

Example 10. Remember that $D_{\text{Id}} \cong \text{Stream}$. For an Id -coalgebra $k : X \rightarrow X$, $k^\alpha = \text{iterate} : X \rightarrow \text{Stream } X$. And obviously, from a function $\text{itf} : X \rightarrow \text{stream } X$ producing a stream of iterated results of a function, one can reproduce the original function itf_α by taking the second element of the stream.

The theorem, its proof, and the examples illustrate the key operational intuition about algebras of free monads: any algebra of a free monad works essentially in an iterative fashion. It is equivalent to the iteration of a plain algebra, which processes (deep) terms by induction on their structure. If we consider a T_F -algebra $\alpha : T_F X \rightarrow X$ as a function reducing a tree with leaves from X into a single X , α_* is the corresponding one-layer function which prescribes how each layer of the tree is collapsed given an

⁷ We are overloading the notation here as a monad is also a functor. The convention is that when T is a monad, \mathbf{C}^T is the category of algebras of the *monad*.

operator and a collection of collapsed subtrees. This intuition is essential for building an intuition about generalized distributive laws, which are to come shortly, and the programs induced by them.

The following fact is useful.

Lemma 2. *Let F and B be endofunctors on \mathbf{C} . Let $\sigma : F \rightarrow \text{Id}$ and $\tau : \text{Id} \rightarrow B$ be natural transformations. Then σ^* and τ° defined point-wise as $(\sigma^*)_X \equiv (\sigma_X)^*$ and $(\tau^\circ)_X \equiv (\tau_X)^\circ$, respectively, are natural transformations $\sigma^* : T_F \rightarrow \text{Id}$, $\tau^\circ : \text{Id} \rightarrow D_B$.*

Proof. Simple, by naturality. □

We continue lifting concepts from § 4.

Lemma 3. *Let 0 be the initial object in \mathbf{C} . Then μ_0 is the initial T -algebra in \mathbf{C}^T . Dually, δ_1 is the terminal D -coalgebra in \mathbf{C}_D .*

Proof. The monad $T : \mathbf{C} \rightarrow \mathbf{C}$ splits into the adjunction $F \dashv U$, where $U : \mathbf{C}^T \rightarrow \mathbf{C}$ is the forgetful functor and $FX = \mu_X$. The conclusion follows from the fact that left adjoints preserve colimits and the initial object is the colimit of the empty diagram. □

We can now lift the definitions of folds and unfolds.

Definition 10. *Let α be a T -algebra. Then $(\downarrow \alpha)_T$ denotes the unique arrow in \mathbf{C} from the initial T -algebra to α . Dually, for a D -coalgebra β and $(\uparrow \beta)_D$.*

$$h = (\downarrow \alpha)_{T_F} \Leftrightarrow h \circ \mu_0 = \alpha \circ Fh, \quad \text{and } \alpha \text{ is a } T\text{-algebra} \quad (22)$$

$$h = (\uparrow \beta)_{D_F} \Leftrightarrow \delta_1 \circ h = Fh \circ \beta, \quad \text{and } \beta \text{ is a } D\text{-coalgebra} \quad (23)$$

The following simple fact complements Theorem 2 by making explicit the isomorphism of (co)initial (co)algebra morphisms.

Lemma 4. *For a T_F -algebra α and a D_B -coalgebra β :*

$$(\downarrow \alpha)_{T_F} = (\downarrow \alpha_*)_F \quad (24)$$

$$(\uparrow \beta)_{D_B} = (\uparrow \beta_\circ)_B \quad (25)$$

Proof. Immediate by Theorem 2. □

6.3 Distributive laws of monads over comonads

Distributive laws of monads over comonads, due to J. Beck [2], are liftings of the plain distributive laws of § 4, which had a straightforward computational interpretation, to monads and comonads. Again, they are natural transformations on the functors, but they respect the additional structure of the monad and comonad in question.

The following is standard, here taken from [36].

Definition 11. Let $\langle T, \eta, \mu \rangle$ be a monad and $\langle D, \varepsilon, \delta \rangle$ be a comonad in a category \mathcal{C} . A *distributive law* of T over D is a natural transformation

$$\Lambda : TD \longrightarrow DT$$

satisfying the following:

$$\Lambda \circ \eta_D = D\eta \tag{26}$$

$$\Lambda \circ \mu_D = D\mu \circ \Lambda_T \circ T\Lambda \tag{27}$$

$$\varepsilon_T \circ \Lambda = T\varepsilon \tag{28}$$

$$\delta_T \circ \Lambda = D\Lambda \circ \Lambda_D \circ T\delta \tag{29}$$

In the following, we relate “programming” to distributive laws, where we further emphasize the view that programs are represented as natural transformations. First, we show that natural transformations with uses of monads and comonads give rise to distributive laws of monads over comonads; see the following theorem. In this manner, we go beyond the simple natural transformations and distributive laws of § 4, and hence we can dualize more programs.

Theorem 3 (Plotkin and Turi, 1997). Let F and B be endofunctors. Natural transformations of type

$$F(\text{Id} \times B) \longrightarrow BT_F \tag{30}$$

or

$$FD_B \longrightarrow B(\text{Id} + F) \tag{31}$$

give rise to distributive laws $\Lambda : T_FD_B \longrightarrow D_B T_F$.

Proof. See [36]. □

The theorem originated in the context of categorical operational semantics [36]; see the related work discussion in § 7. However, the theorem directly applies to our situation of “programming with natural transformations”. In the Haskell-based illustrations of § 5, we encountered such natural transformations as arguments of the enhanced fold operations. We did not exhaust the full generality of the typing scheme that is admitted by the theorem, but we did have occurrences of a free monad and a cofree comonad. Here we note that the general types of natural transformations in the above theorem admit paramorphisms [25] (c.f. $F(\text{Id} \times B)$ in (30)) and their duals, *apomorphisms* [38] (c.f. $B(\text{Id} + F)$ in (31)).

Example 11. The type $F\text{ExprArg}$ in § 5.3 models natural transformations of type $FB \longrightarrow BT_F$ for $F \equiv \text{ExprF}$ and $B \equiv \text{Id}$. Likewise, The type $S\text{ExprArg}$ in § 5.4 models natural transformations of type $FD_B \longrightarrow BF$, for $F \equiv \text{ExprF}$ and $B \equiv \text{Id}$.

Natural transformations $\lambda : FB \longrightarrow BF$ can be lifted so that Theorem 3 applies:

$$B\tau \circ \lambda_{T_F} \circ FB\eta \circ F\pi_2 : F(\text{Id} \times B) \longrightarrow BT_F \tag{32}$$

$$B\iota_2 \circ BF\epsilon \circ \lambda_{D_B} \circ F\xi : FD_B \longrightarrow B(\text{Id} + F) \tag{33}$$

We now give without proof the details of the construction of a distributive law $\bar{\lambda} : TD \longrightarrow DT$ from a natural transformation $\lambda : FB \longrightarrow BF$. Here and in the rest of this section $T \equiv T_F$ and $D \equiv D_B$ for brevity.

$$\begin{aligned}
& \lambda : FB \longrightarrow BF \\
& B\mu \circ B\tau_T \circ \lambda_{T^2} \circ FB\eta_T : FBT \longrightarrow BT \\
& \quad = \{\text{by naturality}\} \\
& B\tau \circ \lambda_T : FBT \longrightarrow BT \\
& (B\tau \circ \lambda_T)^* : TBT \longrightarrow BT \\
& (B\tau_D \circ \lambda_{TD})^* \circ TB\eta_D \circ T\xi : TD \longrightarrow BT \\
& ((B\tau_D \circ \lambda_{TD})^* \circ TB\eta_D \circ T\xi)^\circ : TD \longrightarrow DTD \\
& DT\epsilon \circ ((B\tau_D \circ \lambda_{TD})^* \circ TB\eta_D \circ T\xi)^\circ : TD \longrightarrow DT \tag{34}
\end{aligned}$$

By duality the following has the same type:

$$(D\tau \circ DF\epsilon_T \circ (\lambda_{DT} \circ F\xi_T)^\circ)^* \circ TD\eta : TD \longrightarrow DT \tag{35}$$

One can check directly or by comparison with the construction in [36] that both (34) and (35) define a distributive law. Here we prove in addition that they define the same distributive law.

Lemma 5. *The following holds:*

$$DT\epsilon \circ ((B\tau_D \circ \lambda_{TD})^* \circ TB\eta_D \circ T\xi)^\circ = (D\tau \circ DF\epsilon_T \circ (\lambda_{DT} \circ F\xi_T)^\circ)^* \circ TD\eta$$

In the proof of Lemma 5, the following simple lemma will be useful:

Lemma 6. *For $\varphi : FX \longrightarrow X$, $\psi : X \longrightarrow BX$, $f : Y \longrightarrow X$ and $g : X \longrightarrow Y$ the following holds:*

$$\varphi^* \circ Tf = \llbracket f \nabla \varphi \rrbracket_{F_Y^\dagger} : TY \longrightarrow X \tag{36}$$

$$Dg \circ \psi^\circ = \llbracket g \Delta \psi \rrbracket_{B_X^\ddagger} : X \longrightarrow DY \tag{37}$$

Proof. Simple by fusion. \square

Proof. We can now return to the proof of Lemma 5. By the previous lemma, one of the liftings (34) is equal to

$$\llbracket T\epsilon \Delta \llbracket B\eta_D \circ \xi \nabla B\tau_D \circ \lambda_{TD} \rrbracket_{F_D^\dagger} \rrbracket_{B_T^\ddagger}$$

and the other one (35) is equal to

$$\llbracket D\eta \nabla \llbracket \tau \circ F\epsilon_T \Delta \lambda_{DT} \circ F\xi_T \rrbracket_{B_T^\ddagger} \rrbracket_{F_D^\dagger}$$

We establish the equality by (2) if we can show that

$$\begin{aligned}
& \llbracket T\epsilon \Delta \llbracket B\eta_D \circ \xi \nabla B\tau_D \circ \lambda_{TD} \rrbracket_{F_D^\dagger} \rrbracket_{B_T^\ddagger} \circ \text{in}_{F_D^\dagger} = \\
& (D\eta \nabla \llbracket \tau \circ F\epsilon_T \Delta \lambda_{DT} \circ F\xi_T \rrbracket_{B_T^\ddagger}) \circ F_D^\dagger \llbracket T\epsilon \Delta \llbracket B\eta_D \circ \xi \nabla B\tau_D \circ \lambda_{TD} \rrbracket_{F_D^\dagger} \rrbracket_{B_T^\ddagger}
\end{aligned}$$

We start from the left-hand side as follows (here let $\psi \equiv B\eta_D \circ \xi \nabla B\tau_D \circ \lambda_{TD}$):

$$\begin{aligned}
& \llbracket T\epsilon \Delta (\psi)_{F_D^\dagger} \rrbracket_{B_T^\ddagger} \circ \text{in}_{F_D^\dagger} \\
= & \{ \text{By “AnaFusion” (14)} \\
& (T\epsilon \Delta (\psi)_{F_D^\dagger}) \circ \text{in}_{F_D^\dagger} \\
= & \{ \text{Properties of } \Delta \} \\
& T\epsilon \circ \text{in}_{F_D^\dagger} \Delta (\psi)_{F_D^\dagger} \circ \text{in}_{F_D^\dagger} \\
= & \{ \text{Catamorphism evaluation rule (2), definition of } F_D^\dagger, \text{ definition of } \psi \} \\
& T\epsilon \circ \text{in}_{F_D^\dagger} \Delta (B\eta_D \circ \xi \nabla B\tau_D \circ \lambda_{TD}) \circ (\text{id}_D + F(\psi)_{F_D^\dagger}) \\
= & \{ \text{Properties of } \nabla \} \\
& T\epsilon \circ \text{in}_{F_D^\dagger} \Delta (B\eta_D \nabla B\tau_D) \circ (\xi + \lambda_{TD} \circ F(\psi)_{F_D^\dagger}) \\
= & \{ B\eta \nabla B\tau = \text{Bin} \circ (B\iota_1 \nabla B\iota_2) \} \\
& \text{id}_T \circ T\epsilon \circ \text{in}_{F_D^\dagger} \Delta \text{Bin}_{F_D^\dagger} \circ (B\iota_1 \nabla B\iota_2) \circ (\xi + \lambda_{TD} \circ F(\psi)_{F_D^\dagger}) \\
= & \{ \text{Definition of } B_T^\ddagger \} \\
& B_T^\ddagger \cdot \text{in}_{F_D^\dagger} \circ (T\epsilon \circ \text{in}_{F_D^\dagger} \Delta (B\iota_1 \nabla B\iota_2) \circ (\xi + \lambda_{TD} \circ F(\psi)_{F_D^\dagger})) \\
& \} \\
& \llbracket T\epsilon \circ \text{in}_{F_D^\dagger} \Delta (B\iota_1 \nabla B\iota_2) \circ (\xi + \lambda_{TD} \circ F(\psi)_{F_D^\dagger}) \rrbracket_{B_T^\ddagger}
\end{aligned}$$

The right-hand side simplifies as follows:

$$\begin{aligned}
& (D\eta \nabla \llbracket \tau \circ F\epsilon_T \Delta \lambda_{DT} \circ F\xi_T \rrbracket_{B_T^\ddagger}) \circ F_D^\dagger \llbracket T\epsilon \Delta (\psi)_{F_D^\dagger} \rrbracket_{B_T^\ddagger} \\
= & \{ \text{Definition of } F_D^\dagger \} \\
& D\eta \nabla \llbracket \tau \circ F\epsilon_T \Delta \lambda_{DT} \circ F\xi_T \rrbracket_{B_T^\ddagger} \circ F \llbracket T\epsilon \Delta (\psi)_{F_D^\dagger} \rrbracket_{B_T^\ddagger} \\
= & \{ \text{By “AnaFusion” on the right of } \nabla, \text{ where } \varphi \equiv T\epsilon \Delta (\psi)_{F_D^\dagger} \}
\end{aligned}$$

$$\begin{aligned}
& (\tau \circ F\epsilon_T \Delta \lambda_{DT} \circ F\xi_T) \circ F\llbracket \varphi \rrbracket_{B_T^\ddagger} \\
= & \{ \text{Properties of } \Delta \} \\
& (\tau \times \lambda_{DT}) \circ (F\epsilon_T \Delta F\xi_T) \circ F(\llbracket \varphi \rrbracket_{B_T^\ddagger}) \\
= & \{ \text{definition of out} \} \\
& (\tau \times \lambda_{DT}) \circ (F\pi_1 \Delta F\pi_2) \circ F(\text{out}_{B_T^\ddagger} \circ \llbracket \varphi \rrbracket_{B_T^\ddagger}) \\
= & \{ \text{Anamorphism evaluation rule (4)} \} \\
& (\tau \times \lambda_{DT}) \circ (F\pi_1 \Delta F\pi_2) \circ F(B_T^\ddagger \llbracket \varphi \rrbracket_{B_T^\ddagger} \circ \varphi) \\
= & \{ \text{Properties of } \Delta \} \\
& \tau \circ F(\pi_1 \circ B_T^\ddagger \llbracket \varphi \rrbracket_{B_T^\ddagger} \circ \varphi) \Delta \lambda_{DT} \circ F(\pi_2 \circ B_T^\ddagger \llbracket \varphi \rrbracket_{B_T^\ddagger} \circ \varphi) \\
= & \{ \text{Definition of } B_T^\ddagger \text{ and of } \varphi \} \\
& \tau \circ FT\epsilon \Delta \lambda_{DT} \circ FB\llbracket \varphi \rrbracket_{B_T^\ddagger} \circ F\langle \psi \rangle_{F_D^\dagger} \\
= & \{ \text{Naturality of } \lambda, \text{id}_T \} \\
& \text{id}_T \circ \tau \circ FT\epsilon \Delta BF\llbracket \varphi \rrbracket_{B_T^\ddagger} \circ \lambda_{DT} \circ F\langle \psi \rangle_{F_D^\dagger} \\
= & \{ \text{Definition of } B_T^\ddagger \text{ and of } \varphi, \text{definition of } \varphi \} \\
& B_T^\ddagger F\llbracket T\epsilon \Delta \langle \psi \rangle_{F_D^\dagger} \rrbracket_{B_T^\ddagger} \circ (\tau \circ FT\epsilon \Delta \lambda_{TD} \circ F\langle \psi \rangle_{F_D^\dagger}) \\
& D\eta \nabla \llbracket \tau \circ FT\epsilon \Delta \lambda_{TD} \circ F\langle \psi \rangle_{F_D^\dagger} \rrbracket_{B_T^\ddagger}
\end{aligned}$$

It thus remains to show that:

$$\llbracket T\epsilon \circ \text{in}_{F_D^\dagger} \Delta (B\iota_1 \nabla B\iota_2) \circ (\xi + \lambda_{TD} \circ F\langle \psi \rangle_{F_D^\dagger}) \rrbracket_{B_T^\ddagger} = D\eta \nabla \llbracket \tau \circ FT\epsilon \Delta \lambda_{TD} \circ F\langle \psi \rangle_{F_D^\dagger} \rrbracket_{B_T^\ddagger}$$

This can be done by the universal property of anamorphisms (4), where $\omega \equiv \tau \circ FT\epsilon \Delta \lambda_{TD} \circ F\langle \psi \rangle_{F_D^\dagger}$:

$$\begin{aligned}
& \text{out}_{B_T^\ddagger} \circ (D\eta \nabla \llbracket \omega \rrbracket_{B_T^\ddagger}) \\
= & \{ \text{Properties of } \nabla \} \\
& \text{out}_{B_T^\ddagger} \circ D\eta \nabla \text{out}_{B_T^\ddagger} \circ \llbracket \omega \rrbracket_{B_T^\ddagger} \\
= & \{ (4) \} \\
& \text{out}_{B_T^\ddagger} \circ D\eta \nabla B_T^\ddagger \llbracket \omega \rrbracket_{B_T^\ddagger} \circ \omega \\
= & \{ \text{Definitions of out, } B_T^\ddagger \text{ and } \omega, \text{ and properties of } \Delta \}
\end{aligned}$$

$$\begin{aligned}
& (\epsilon_T \circ D\eta \triangle \xi_T \circ D\eta) \nabla (\tau \circ FT\epsilon \triangle B[\omega]_{B_T^\ddagger} \circ \lambda_{TD} \circ F(\psi)_{F_D^\dagger}) \\
= & \{ \text{“Abides law”} \} \\
& (\epsilon_T \circ D\eta \nabla \tau \circ FT\epsilon) \triangle (\xi_T \circ D\eta \nabla B[\omega]_{B_T^\ddagger} \circ \lambda_{TD} \circ F(\psi)_{F_D^\dagger}) \\
= & \{ \text{Naturality:} \\
& \quad \epsilon_T \circ D\eta = T\epsilon \circ \eta_D \\
& \quad \tau \circ FT\epsilon = T\epsilon \circ \tau_D \\
& \quad \xi_T \circ D\eta = BD\eta \circ \xi \} \\
& (T\epsilon \circ \eta_D \nabla T\epsilon \circ \tau_D) \triangle (BD\eta \circ \xi \nabla B[\omega]_{B_T^\ddagger} \circ \lambda_{TD} \circ F(\psi)_{F_D^\dagger}) \\
= & \{ \text{Definition of in}_{F_D^\dagger} \} \\
& T\epsilon \circ \text{in}_{F_D^\dagger} \triangle (BD\eta \circ \xi \nabla B[\omega]_{B_T^\ddagger} \circ \lambda_{TD} \circ F(\psi)_{F_D^\dagger}) \\
= & \{ \text{Properties of } \triangle, \text{ properties of } \nabla, Bf \nabla Bg = B(f \nabla g) \circ (B\iota_1 \nabla B\iota_2) \} \\
& (\text{id}_T \times B(D\eta \nabla [\omega]_{B_T^\ddagger})) \circ (T\epsilon \circ \text{in}_{F_D^\dagger} \triangle (B\iota_1 \nabla B\iota_2)) \circ (\xi + \lambda_{TD} \circ F(\psi)_{F_D^\dagger}) \\
= & \{ \text{Definition of } B_T^\ddagger \text{ and } \omega \} \\
& B_T^\ddagger(D\eta \nabla [\tau \circ FT\epsilon \triangle \lambda_{TD} \circ F(\psi)_{F_D^\dagger}]_{B_T^\ddagger}) \circ \\
& (T\epsilon \circ \text{in}_{F_D^\dagger} \triangle (B\iota_1 \nabla B\iota_2)) \circ (\xi + \lambda_{TD} \circ F(\psi)_{F_D^\dagger})
\end{aligned}$$

□

The following definition is therefore well-formed.

Definition 12. For a natural transformation $\lambda : FB \longrightarrow BF$, we denote by $\bar{\lambda}$ the distributive law $T_F D_B \longrightarrow D_B T_F$ given either by (34) or (35).

6.4 Conservativeness of free distributive laws

We can lift simple distributive laws of § 4 to distributive laws of monads over comonads. It remains to establish that such a lifting of distributive laws is semantics-preserving. The gory details follow.

In the simple development of § 4, we constructed algebras and coalgebras from distributive laws by simple projections and injections; c.f. equations (10) and (7). These constructions are lifted as follows:

Lemma 7. For all X in \mathbf{C} , the arrow

$$D\mu_X \circ \Lambda_{TX} : TDTX \longrightarrow DTX \quad (38)$$

is a T -algebra. Dually, the arrow

$$\Lambda_{DX} \circ T\delta_X : TDX \longrightarrow DTDX \quad (39)$$

is a D -coalgebra.

Proof. We must verify that the two T -algebra laws (20) and (21) hold. This can be done by a simple calculation involving just naturality and definitions. We give here just the proof of (21), (20) is even simpler. The dual part of the lemma follows by duality.

$$\begin{aligned}
& D\mu \circ \Lambda_T \circ \mu_{DT} \\
= & \{ \text{by (27)} \} \\
& D\mu \circ D\mu_T \circ \Lambda_{T^2} \circ T\Lambda_T \\
= & \{ \text{monad laws} \} \quad \square \\
& D\mu \circ DT\mu \circ \Lambda_{T^2} \circ T\Lambda_T \\
= & \{ \Lambda \text{ is a natural transformation} \} \\
& D\mu \circ \Lambda_T \circ TD\mu \circ T\Lambda_T
\end{aligned}$$

Now (38) is a T -algebra, and thus by Lemma 3 and Def. 10 it induces an arrow:

$$\langle D\mu_0 \circ \Lambda_{T0} \rangle_T : T0 \longrightarrow DT0 \quad (40)$$

Moreover, when T and D are free on F and B respectively, this is equivalent to

$$\langle D\mu_0 \circ \Lambda_{T0} \rangle_T : \mu F \longrightarrow D\mu F$$

which is by Theorem 2 isomorphic to a B -coalgebra

$$\langle D\mu_0 \circ \Lambda_{T0} \rangle_{T_F \circ} : \mu F \longrightarrow B\mu F \quad (41)$$

Dually for (39):

$$\llbracket \Lambda_{D1} \circ T\delta_1 \rrbracket_D : TD1 \longrightarrow D1 \quad (42)$$

$$\llbracket \Lambda_{D1} \circ T\delta_X \rrbracket_{D_{B^*}} : F\nu B \longrightarrow \nu B \quad (43)$$

Compare with equations (11) and (8). This shows that *any* distributive law of a free monad over a cofree comonad also gives rise to an algebra for a catamorphisms and a coalgebra for object construction.

Example 12. The functions `sFoldExpr` in § 5.4, and `fFoldExpr` in § 5.3 are examples of (41) where B in (41) is fixed to be the identity functor.

The following theorem establishes the essential property that the (co)monadic development of the present section entails the development of § 4.

Theorem 4. *Let F and B be endofunctors on a category \mathbf{C} . Let $\lambda : FB \longrightarrow BF$ be a natural transformation. Then the following holds.*

$$\llbracket \lambda_{\nu B} \circ F\text{out}_B \rrbracket_B = \llbracket \bar{\lambda}_{D_{B1}} \circ T_F\delta_X \rrbracket_{D_{B^*}} : F\nu B \longrightarrow \nu B \quad (44)$$

$$\langle \text{Bin}_F \circ \lambda_{\mu F} \rangle_F = \langle D_B\mu_0 \circ \bar{\lambda}_{T_F0} \rangle_{T_F \circ} : \mu F \longrightarrow B\mu F \quad (45)$$

In order to prove Theorem 4, we will need a few simple facts. First, we repeat the following important definitions from the proof of Theorem 2. Here, $\varphi : FX \rightarrow X$, $\psi : X \rightarrow BX$, $f : TX \rightarrow Y$, $g : Y \rightarrow DX$; $\eta, \tau, \epsilon, \xi$ are defined by the free monad $T \equiv T_F$ and the cofree comonad $D \equiv D_B$. Note carefully the types of f and g respectively: we are extending the domain of operations $*$ and \circlearrowleft to any arrows out of the free monad and into the cofree comonad, respectively.

$$f_* \equiv f \circ \tau_X \circ F\eta_X \quad (46)$$

$$\varphi^* \equiv (\text{id}_X \nabla \varphi)_{F_X^\dagger} \quad (47)$$

$$g_\circlearrowleft \equiv B\epsilon_X \circ \xi_X \circ g \quad (48)$$

$$\psi^\circlearrowleft \equiv (\text{id}_X \Delta \psi)_{B_X^\ddagger} \quad (49)$$

Lemma 8. *Operations $*$ and \circlearrowleft are commutative. Namely, whenever the types make sense:*

$$(\alpha_\circlearrowleft)_* = (\alpha_*)_\circlearrowleft$$

Proof. Trivial. □

We therefore omit brackets in nested applications of the above operators and in the following text write just $\alpha_{*\circlearrowleft}$ or $\alpha_{\circlearrowleft*}$.

Lemma 9. *For $f : TY \rightarrow Z$, $h : Z \rightarrow DX$ and $g : X \rightarrow Y$, $k : Z \rightarrow Z'$, $l : Z' \rightarrow Z$:*

$$(k \circ f \circ Tg)_* = k \circ f_* \circ Fg \quad (50)$$

$$(Dg \circ h \circ l)_\circlearrowleft = Bg \circ h_\circlearrowleft \circ l \quad (51)$$

Proof. For (50) we calculate as follows:

$$\begin{aligned} & (k \circ f \circ Tg)_* \\ &= \{ \text{Definition of } * \text{ (46)} \} \\ & \quad k \circ f \circ Tg \circ \tau_X \circ F\eta_X \\ &= \{ \text{naturality of } \tau \circ F\eta : F \rightarrow T \} \\ & \quad k \circ f \circ \tau_Y \circ F\eta_Y \circ Fg \\ &= \{ \text{Definition of } * \text{ (46)} \} \\ & \quad k \circ f_* \circ Fg \end{aligned}$$

The other equation (51) follows by duality. □

Lemma 10. For endofunctors F , B , $\langle T, \mu, \eta \rangle$ and $\langle D, \delta, \epsilon \rangle$ as before, the following holds:

$$\begin{aligned}
\mu_0 &= \text{in}_F^* \\
\mu_{0*} &= \text{in}_F \\
\delta_1 &= \text{out}_B^\infty \\
\delta_{1\infty} &= \text{out}_B
\end{aligned} \tag{52}$$

Proof. Immediate by Theorem 2 and Lemma 3. \square

We can now proceed with the proof of Theorem 4.

Proof. By Definition 12, $\bar{\lambda}$ is defined as follows:

$$\bar{\lambda} = DT\epsilon \circ ((B\tau_D \circ \lambda_{TD})^* \circ TB\eta_D \circ T\xi)^\infty : TD \longrightarrow DT \tag{53}$$

Then the following calculation establishes equality (45).

$$\begin{aligned}
& \langle D\mu_0 \circ \bar{\lambda}_{T0} \rangle_{T\infty} \\
&= \{ \text{Lemma 4} \} \\
& \langle (D\mu_0 \circ \bar{\lambda}_{T0})^* \rangle_{F\infty} \\
&= \{ \text{Definition (??) of } \infty \} \\
& B\epsilon_{T0} \circ \xi_{T0} \circ \langle (D\mu_0 \circ \bar{\lambda}_{T0})_* \rangle_F \\
&= \{ \text{By ‘‘CataFusion’’ (15):} \\
& \quad B\epsilon_T \circ \xi_T \circ (D\mu \circ \bar{\lambda}_T)_* \\
&= \{ \text{Definition (48) of } \infty \} \\
& \quad (D\mu \circ \bar{\lambda}_T)_{*\infty} \\
&= \{ \text{Definition (53) of } \bar{\lambda} \} \\
& \quad (D\mu \circ DT\epsilon_T \circ ((B\tau_{DT} \circ \lambda_{TDT})^* \circ TB\eta_{DT} \circ T\xi_T)^\infty)_{*\infty} \\
&= \{ (51): ‘‘(Dg \circ h)_\infty = Bg \circ h_\infty’’ \} \\
& \quad (B\mu \circ BT\epsilon_T \circ ((B\tau_{DT} \circ \lambda_{TDT})^* \circ TB\eta_{DT} \circ T\xi_T)^\infty)_{*\infty} \\
&= \{ \text{Theorem 2: } \infty \text{ and } \infty \text{ are inverse} \} \\
& \quad (B\mu \circ BT\epsilon_T \circ (B\tau_{DT} \circ \lambda_{TDT})^* \circ TB\eta_{DT} \circ T\xi_T)_* \\
&= \{ \text{Naturality: } (B\tau \circ \lambda_T)^* \circ TB\eta : TB \longrightarrow BT \} \\
& \quad (B\mu \circ (B\tau_T \circ \lambda_{T^2})^* \circ TB\eta_T \circ TB\epsilon_T \circ T\xi_T)_* \\
&= \{ \text{Lemma 9} \} \\
& \quad (B\mu \circ (B\tau_T \circ \lambda_{T^2})^* \circ TB\eta_T)_* \circ F(B\epsilon_T \circ \xi_T) \}
\end{aligned}$$

$$\begin{aligned}
& \langle (B\mu_0 \circ (B\tau_{T0} \circ \lambda_{T^2_0})^* \circ TB\eta_{T0})_* \rangle_F \\
= & \{ \text{Lemma 9: } "(k \circ f \circ Tg)_* = k \circ f_* \circ Fg" \} \\
& \langle B\mu_0 \circ (B\tau_{T0} \circ \lambda_{T^2_0})_* \circ FB\eta_{T0} \rangle_F \\
= & \{ \text{Theorem 2: } * \text{ and } * \text{ are inverse} \} \\
& \langle B\mu_0 \circ B\tau_{T0} \circ \lambda_{T^2_0} \circ FB\eta_{T0} \rangle_F \\
= & \{ \text{naturality of } \lambda : FB \longrightarrow BF \} \\
& \langle B\mu_0 \circ B\tau_{T0} \circ BF\eta_{T0} \circ \lambda_{T0} \rangle_F \\
= & \{ \text{definition (46) of } * \} \\
& \langle B(\mu_{0*}) \circ \lambda_{T0} \rangle_F \\
= & \{ (52) \} \\
& \langle Bin_F \circ \lambda_T \rangle_F
\end{aligned}$$

Equation (44) follows by duality. \square

6.5 The generalized expression lemma

It remains to lift Theorem 1 (the “simple expression lemma”). As a preparation, we need an analog of the fusion rule.

Lemma 11. *For D -coalgebras α and β :*

$$\langle \langle \alpha \rangle_D \circ f \rangle = \langle \langle \beta \rangle_D \rangle \iff \alpha \circ f = Df \circ \beta \quad (54)$$

Proof. Immediate by uniqueness of the terminal morphism, as before. \square

Theorem 5 (“Generalized expression lemma”). *Let $\langle T, \eta, \mu \rangle$ be a monad and $\langle D, \eta, \delta \rangle$ be a comonad. Let $\Lambda : TD \longrightarrow DT$ be a distributive law of the monad T over D . Then the following holds:*

$$\langle \langle \Lambda_{D1} \circ T\delta_1 \rangle_D \rangle_T = \langle \langle \langle D\mu_0 \circ \Lambda_{T0} \rangle_T \rangle_D \rangle$$

Proof. The proof has exactly the same structure as that of Theorem 1 except that we have to check at all relevant places that the algebras and coalgebras in question satisfy the additional properties (20) and (21) or their duals, subject to straightforward applications of the monad laws, properties (26) - (29) of distributive laws, and by naturality. We give an outline of the proof of the theorem while omitting the routine checks.

$$\begin{aligned}
& \langle \langle \Lambda_D \circ T\delta \rangle_D \rangle_T \circ T \langle \langle D\mu \circ \Lambda_T \rangle_T \rangle_D \\
= & \{ \text{By (54)} \} \\
& \langle \langle \Lambda_D \circ T \langle \langle D\mu \circ \Lambda_T \rangle_T \rangle_D \rangle_D \rangle \\
= & \{ \text{By (54)} \} \\
& \langle \langle \langle D\mu \circ \Lambda_T \rangle_T \rangle_D \rangle \circ \mu
\end{aligned}$$

The conclusion follows by (22). \square

7 Related work

Functional OO programming

We are not aware of any similar treatment of the correspondence between functional and OO programming. Initially, one would expect some previous work on functional OO programming to be relevant here, such as Moby [12] (an ML-like language with a class mechanism), C# 3.0/VB 9.0/LINQ [3] (the latest .NET languages that incorporate higher-order list-processing functions and more type inference), F# (an ML/OCaml-inspired language that is married with .NET objects), Scala [29,8] (a Java-derived language with support for functional programming), ML-ART or OCaml [32] (ML with OO-targeting type extensions), O'Haskell [28] (an extension of Haskell with subtyping), or various language extension proposals, e.g., for pattern matching [33,9,10]. However, all such work has not revealed the expression lemma. *When functional OO efforts start from a functional language*, then the focus is normally on type-system extensions for subtyping, self, and inheritance, while OO programs are essentially encoded as functional programs, without though relating the encoding results to any “native” functional counterparts. *Dually, when functional OO efforts start from an OO language*, then the focus is normally on translations that eliminate functional idioms, without though relating the translation results to any “native” OO counterpart. (For instance, Scala essentially models a function as a special kind of object.) Our approach specifically leverages the correspondence between functional folds and OO designs based on an idealized composite pattern. We believe that our work may turn out to be useful as a foundation for refactorings in functional OO languages.

The expression problem

Previous work on the expression problem [40,43,34,24,23] has at best assumed the expression lemma implicitly. The lemma may have been missing because the expression problem classically assumes only very little structure: essentially, there are supposed to be multiple data variants as well as multiple operations on these variants. In contrast, the proposed expression lemma requires more structure, i.e., it requires functional folds or OO designs based on an idealized composite pattern, respectively. It will be interesting to generalize the given expression lemma further.

Programming vs. semantics

We have demonstrated how distributive laws of a functor over a functor (both possibly with additional structure) arise naturally from programming practice with disciplined folds and an idealized composite pattern. By abstraction, we have ultimately arrived at the same notion of adequacy that Turi and Plotkin originally coined for denotational and operational semantics [35,36]. There, our functional programming side of the picture corresponds to *denotational semantics* and the OO programming side corresponds to *operational semantics*. Our functional/OO programming correspondence corresponds to *adequacy of denotational and operational semantics*. We have provided a simple, alternative, calculational proof geared towards functional programming intuitions. Any

further correspondence, for instance of their *operational rules*, is not straightforward. We hypothesize that an elaborated expression lemma may eventually incorporate additional structure that has no direct correspondence in Turi and Plotkin’s sense.

More on distributive laws

Distributive laws of a functor over a functor (both possibly with additional structure) [2] have recently enjoyed renewed interest. We mention a few of the more relevant contributions. In the context of bialgebraic semantics, Fiore, Plotkin and Turi have worked on languages with binders in a presheaf category [11], and Bartek Klin has worked on recursive constructs [21]. Both theoretical contributions may inspire a model of object structures with cycles and sharing in our interpretation. Modular constructions on distributive laws, including those we leveraged towards the end of § 4.3 have been investigated by Bart Jacobs [17]. More advanced modular constructions may be helpful in the further exploration of the modularity of dualizable programs. Alberto Pardo, Tarmo Uustalu, Varmo Vene, and collaborators have been using distributive laws for recursion and corecursion schemes. For instance, in [37], a generalized coinduction scheme is delivered where a distributive law specifies the pattern of mutual recursion between several functions defined by coinduction. This work seems to be related to coalgebraic OO programming where methods in an interface are (possibly) mutually recursive.

8 Concluding remarks

We have revealed the expression lemma — a correspondence between OO and functional programs, subject to the assumption that both kinds of programs share a certain decomposition based on structural recursion. The decomposition requirement for functional programs is equivalent to a class of natural folds. The decomposition requirement for OO programs is equivalent to the concept of object structures with part-whole relationships and methods that are in alignment with an idealized composite design pattern. The formal development for comparing functional and OO programs relies on a coalgebraic model of functional objects.

While our development already covers some non-trivial idioms in “dualizable” programming, e.g., iteration of term construction and recursive function application, it still leaves many open questions — in particular, if we wanted to leverage the duality for real-world programs. Hence, one challenge is to generalize the expression lemma and the associated constructions of distributive laws so that the class of dualizable programs is extended. For instance, histomorphisms [37,19] are not just useful in devising efficient encodings for non-linearly recursive problems, they also generalize access to intermediate results for non-immediate components — very much in the sense of “dotting” into objects and invoking methods on non-immediate components. Another challenge is to admit data structures with sharing and cycles. The use of sharing and cycles is common in OO programming — even without the additional complication of mutable objects. Yet another challenge is to complete the current understanding of the functional/OO correspondence into effective bidirectional refactorings. For instance, in

the objects-to-functions direction, such a refactoring would involve non-trivial preconditions on the shape of the OO code, e.g., preconditions to establish absence of sharing, cycles, and mutations, where we may be able to leverage related OO type-system extensions, e.g., for immutability and ownership [7,4]. More generally, we seek a *calculational* framework in which we can represent and relate between the various entities of our approach: algebras, coalgebras, distributive laws, as well as freewheeling functional and OO programs. There is a huge body of prior art of calculation for recursive programs [5,22,15,16,6].

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