Exercise 1. Let $F$ be the following formula:

$$F = \neg((\neg(P \land \neg Q)) \lor (R \lor \neg S)) \lor (U \land V)$$

(1) Compute the negation normal form (NNF) $F'$ of $F$.

(2) Convert $F'$ to CNF using the satisfiability-preserving transformation described in the lecture.

Exercise 2. Let $N$ be the following set of clauses:

1. $P_1 \lor P_3$
2. $P_1 \lor P_3 \lor P_3$
3. $\neg P_3 \lor \neg P_4$
4. $P_4 \lor P_5$
5. $P_4 \lor P_5 \lor P_5$

(1) Let $\succ$ be the ordering on propositional variables defined by $P_5 \succ P_4 \succ P_3 \succ P_2 \succ P_1$.
   - Sort the clauses in $N$ according to $\succ_C$.
   - Which literals are maximal in the clauses of $N$?
   - Is the set $N$ of clauses saturated w.r.t. $\text{Res}_S$?
   - Construct a model of $N$ using the canonical construction presented in the lecture.

(2) Let $\succ$ be the ordering on propositional variables defined by $P_5 \succ P_3 \succ P_4 \succ P_2 \succ P_1$.
   - Sort the clauses in $N$ according to $\succ_C$.
   - Which literals are maximal in the clauses of $N$?
   - Is the set $N$ of clauses saturated w.r.t. $\text{Res}_S$?
     If not, define a selection function $S$ such that $N$ is saturated under $\text{Res}_S$.
   - Construct a model of $N$ using the canonical construction presented in the lecture.

Exercise 3. Prove or refute the following statements:

(a) If $F$ is a first-order formula, then $F$ is valid if and only if $F \rightarrow \bot$ is unsatisfiable.
(b) If $F$ is a first-order formula and $x$ a variable, then $F$ is unsatisfiable if and only if $\exists x F$ is unsatisfiable.

(c) If $F$ and $G$ are first-order formulae, $F$ is valid, and $F \rightarrow G$ is valid, then $G$ is valid.

(d) If $F$ and $G$ are first-order formulae, $F$ is satisfiable, and $F \rightarrow G$ is satisfiable, then $G$ is satisfiable.

(e) If $F$ and $G$ are first-order formulae and $x$ is a variable then $\forall x (F \land G) \models \forall x F \land \forall x G$ and $\forall x F \land \forall x G \models \forall x (F \land G)$.

(f) If $F$ and $G$ are first-order formulae and $x$ is a variable then $\exists x (F \land G) \models \exists x F \land \exists x G$ and $\exists x F \land \exists x G \models \exists x (F \land G)$.

**Exercise 4.** Prove or refute the following statement:
If $t, s, s'$ are terms and $x$ and $y$ are distinct variables, then $(t[s/x])[s'/y] = t[s/x, s'/y]$.

**Exercise 5.** Prove or refute the following statements:

(a) If $F$ and $G$ are first-order formulae and $F \models G$, then $F \models \neg G$ does not hold.

(b) If $F$ and $G$ are first-order formulae and $F \models G$, then $\neg F \models G$ does not hold.

(c) If $F$, $G$, and $H$ are first-order formulae and $F \land G \models H$, then $F \models H$.

(d) If $F$, $G$, and $H$ are first-order formulae and $F \lor G \models H$, then $F \models H$.

(e) If $F$ and $G$ are first-order formulae then if $F$ and $G$ are satisfiable then $F \land G$ is satisfiable.

**Exercise 7.** Compute a most general unifier of

1. $\{ f(x, g(x)) = y, h(y) = h(v), v = f(g(z), x) \}$
2. $\{ f(x, x) = y, h(y) = h(v), v = f(g(z), h(y)) \}$
3. $\{ f(x, g(z)) = y, h(y) = h(v), v = f(g(z), w) \}$

**Exercise 8.** What is the clausal normal form of

$$\exists x \forall x', \exists y ((\forall z (p(x, y, z) \land (x \approx x'))) \rightarrow (\exists u \forall z q(u, y, z) \land \neg r(x, x', y)))$$

**Exercise 9.** Consider the following formulae:

- $F_1 := \forall x (S(x) \rightarrow \exists y (R(x, y) \land P(y)))$
- $F_2 := \forall x (P(x) \rightarrow Q(x))$
- $F_3 := \exists x S(x)$
- $G := \exists x \exists y (R(x, y) \land Q(y))$

Use resolution to prove that $\{F_1, F_2, F_3\} \models G$.

**Exercise 10.** Check the satisfiability of the following formulae using the DAG version of the Congruence Closure algorithm presented in the class:
Exercise 11: Check the satisfiability over \( \mathbb{Z} \) of the following sets of constraints in difference logic:

(1) \( x - y \leq 5 \land y - u \leq 4 \land x - z \leq -1 \land z - x \leq 1 \).

(2) \( x - y \leq 5 \land y - u \leq 4 \land x - z \leq -1 \land z - x \leq 1 \land y - z \leq -7 \).

Hint: It is sufficient to check the existence of negative cycles in \( G(\phi_i) \) by looking at the graphs; in this assignment you do not have to use the Bellman-Ford algorithm for this.

Exercise 12. Let \( F \) be the following conjunction (in linear rational arithmetic):

\[
F: \quad \begin{align*}
x_1 + x_2 + 4x_3 &= 4 \quad \land \\
x_1 + 2x_3 + \frac{1}{5} &< 0 \quad \land \\
x_2 - x_3 &\leq \frac{1}{2} \quad \land \\
x_1 + 6x_3 &\leq \frac{5}{2}
\end{align*}
\]

Check the satisfiability of \( F \) using:

(1) the Fourier-Motzking method for quantifier elimination;

(2) the Loos-Weispfenning method for quantifier elimination.

Exercise 13. Use the Nelson-Oppen procedure for checking the satisfiability of the following formulae:

(1) \( 1 \leq x \land x \leq 3 \land f(x) \neq f(1) \land f(x) \neq f(3) \land f(1) \neq f(2) \)

in the combination \( LI(\mathbb{Z}) \cup U1F(f) \).

(2) \( f(x) \approx x + y \land x \leq y + z \land x + z \leq y \land y = 1 \land f(x) \neq f(2) \)

in the combination \( LI(\mathbb{Z}) \cup U1F(f) \).

(3) \( x + y \approx z \land f(z) \approx z \land f(x + y) \neq z \)

in the combination \( LI(\mathbb{Q}) \cup U1F(f) \).

Exercise 14. Check the satisfiability w.r.t. \( T = LI(\mathbb{Q}) \) of the following set of ground clauses using one of the versions of the \( DPLL(T) \) algorithm presented in the class.

\( (\neg(0 \leq x+1) \lor \neg(3y \leq z)) \land (\neg(z \leq x+3y+1) \lor (3y \leq z)) \land (\neg(0 \leq y) \lor (0 \leq x+1)) \land (z \leq x+1+2y) \)

For theory reasoning in \( LI(\mathbb{Q}) \) use the Fourier-Motzkin algorithm.