Collection of exercises

Exercise 1. Assume $R \succ Q \succ P$. Let $N_1$ be the following set of clauses:

\begin{align*}
(C_1) & \quad \neg R \lor \neg P \\
(C_2) & \quad Q \lor P \\
(C_3) & \quad \neg Q \\
(C_4) & \quad R \lor \neg P \lor Q
\end{align*}

Use the ordered resolution calculus $\text{Res} \succ$ described in the lecture for checking the satisfiability of the set $N_1$ of clauses.

Exercise 2. Assume $P \succ Q \succ R \succ S$. Let $N_2$ be the following set of clauses:

\begin{align*}
(C_1) & \quad \neg Q \lor P \\
(C_2) & \quad R \lor \neg P \\
(C_3) & \quad Q \lor \neg S \\
(C_4) & \quad \neg Q \lor S
\end{align*}

(1) Define a selection function $S$ such that this set of clauses is saturated w.r.t. the ordered resolution calculus with selection $\text{Res}_S \succ$. Justify your choice.

(2) Sort the clauses according to $\succ C$.

(3) Construct a model of $N_2$ using the canonical construction presented in the lecture.

Exercise 3. Give the definition of redundancy of a clause w.r.t. a set $N$ of clauses.

Assume $P \succ Q \succ R \succ S$.

(1) Is the clause $P \lor \neg S$ redundant w.r.t. the set of clauses $\{\neg Q \lor P, R \lor \neg P, Q \lor \neg S\}$?

(2) Is the clause $\neg Q \lor R$ redundant w.r.t. the set of clauses $\{\neg Q \lor P, R \lor \neg P, Q \lor \neg S\}$?

Justify your answers.

Exercise 4. Let $\Sigma = (\{f/1, g/1, h/1, a\}, \{p/2, q/1, r/2\})$. Let $X$ be a set of variables, and assume that $\{x, y, z, u, v, w, s, t\} \subseteq X$.

Let $\succ$ an ordering on ground atoms with the property that for all ground terms $t_1, ..., t_{12}$, $\neg p(t_1, t_2) \succ p(t_3, t_4) \succ \neg q(t_5, t_6) \succ q(t_7, t_8) \succ \neg r(t_9, t_{10}) \succ r(t_{11}, t_{12})$.

Let $N$ be the following set of clauses:

\begin{align*}
(1) & \quad \neg r(f(x), y) \lor p(g(x), x) \\
(2) & \quad \neg q(h(g(z))) \lor \neg p(z, u) \\
(3) & \quad q(h(v)) \\
(4) & \quad r(w, g(s)) \lor p(t, f(s))
\end{align*}
Use the ordered resolution calculus $Res^>$ described in the lecture for checking the satisfiability of the set $N$ of clauses.

**Exercise 5.** Consider the following formulae over a signature containing function symbols $\Omega = \{c/0, f/1\}$ and predicate symbols $\Pi = \{P/1\}$:

- $F_1 := P(c)$
- $F_2 := \forall x(P(x) \rightarrow P(f(x)))$
- $F_3 := P(f(f(c)))$.

Use resolution to prove that $\{F_1, F_2\} \models F_3$.

**Exercise 6.**

(a) Give definitions for the following fragments of first-order logic:
- The Bernays-Schönfinkel class;
- The Ackermann class.

(b) What is the idea in the proof of decidability for the Bernays-Schönfinkel class?

(c) To which of these classes do the following formulae belong (note that they can be in more than one, or in none of the classes above):

1. $\exists y \forall x ((p(x) \lor r(x, y)) \land q(y))$
2. $\forall x \exists y \exists z \exists u ((p(x) \lor q(y)) \land (q(y) \lor p(u)))$
3. $\exists z \forall x \exists y (p(x) \lor q(y)) \land q(z)$

**Exercise 7.** Check the satisfiability of the following formulae using the DAG version of the Congruence Closure algorithm presented in the class:

1. $f(a, b) \approx f(b, a) \land f(c, a) \neq f(b, c)$
2. $f(g(a)) \approx g(f(a)) \land f(g(f(b))) \approx a \land f(b) \approx a \land g(f(a)) \neq a$
3. $f(f(f(a))) \approx f(a) \land f(f(a)) \approx a \land f(a) \neq a$

**Exercise 8.**

1. Check the satisfiability over $\mathbb{Z}$ of the following set of constraints in positive difference logic using the method presented in the lecture. In case of satisfiability find a satisfiable assignment.

   - (a) $x - y \leq 4 \land y - z \leq 2 \land x - z \leq 2 \land z - x \leq -3$
   - (b) $x - y \leq 4 \land y - z \leq 0 \land x - z \leq 4 \land z - x \leq -3 \land x - u \leq -4$

2. Check the satisfiability over $\mathbb{Q}$ of the following sets of constraints:

   - (3) $x - y \leq 5 \land y - u \leq 4 \land x - z \leq -1 \land z - x \leq 1$
   - (4) $x - y \leq 5 \land y - u \leq 4 \land x - z \leq -1 \land z - x \leq 1 \land z - y \leq -5$. 

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Exercise 9. Use the Nelson-Oppen procedure for checking the satisfiability of the following formulae:

1. \( 1 \leq c \land c \leq 3 \land f(c) \neq f(1) \land f(c) \neq f(3) \land f(1) \neq f(2) \)
   in the combination \( LI(\mathbb{Z}) \cup UIF_f \).

2. \( f(c) \approx c + d \land c \leq d + e \land c + e \leq d \land d = 1 \land f(e) \neq f(2) \)
   in the combination \( LI(\mathbb{Z}) \cup UIF_f \).

3. \( c + d \approx e \land f(e) \approx e \land f(c + d) \neq e \)
   in the combination \( LI(\mathbb{Q}) \cup UIF_f \).

Exercise 10. Let \( T = LI(\mathbb{Q}) \), and let \( Q := x \geq 1, R := x \leq y, P := x + x \leq 2 \). Use a DPLL(\( T \)) method to check the satisfiability w.r.t. \( T \) of the following set of clauses:

\[
\begin{align*}
(C_1) & \quad \neg R \lor P \\
(C_2) & \quad \neg Q \lor \neg P \\
(C_4) & \quad R \lor P
\end{align*}
\]

Exercise 11. In what follows we consider the theory of arrays defined in the lecture. We assume that the theory of indices \( T^i \) is \( LI(\mathbb{Z}) \), and the theory of elements \( T^e \) is \( LI(\mathbb{Q}) \).

Which of the formulae below are in the array property fragment and which are not? Justify your answer. (The universally quantified variables \( i, j \) are of sort index; the indices \( k, l, u, i = 1, 2 \) which are not universally quantified are considered to be constants of sort index)

1. \( \forall i \ (a[a[i]] > a[i]) \)
2. \( \forall i \ (i > a[i]) \)
3. \( \forall i \ (a[i] > b[i]) \)
4. \( \forall i \ (i \leq a[k] \rightarrow a[i] = a[k]) \)
5. \( \forall i, j \ (l_1 < i < u_1 < l_2 < j < u_2 \rightarrow a[i] \leq a[j]) \)
6. \( \forall i, j \ (l_1 < i < j < u_2 \rightarrow a[i] \leq a[j]) \)
7. \( \forall i, j \ (l_1 < i \leq j < u_2 \rightarrow a[i] \leq a[j]) \)

Exercise 12. Consider the array property formula:

\[ F : write(a, l, v)[k] = b[k] \land b[k] \neq v \land a[k] = v \land \forall i \ (i \neq l \rightarrow a[i] = b[i]) \]

and let \( F'_6 \) be the formula obtained (in the example presented in the lecture) by applying Steps 1–6 to \( F \), after simplification.

\[ F'_6 : \quad a'[k] = b[k] \land b[k] \neq v \land a[k] = v \land a[\lambda] = b[\lambda] \land (k \neq l \rightarrow a[k] = b[k]) \land a'[l] = v \land a[\lambda] = a'[\lambda] \land (k \neq l \rightarrow a[k] = a'[k]) \land \lambda \neq k \land \lambda \neq l. \]

Check the satisfiability of \( F'_6 \) w.r.t. \( T = UIF_{[a,b,a']} \cup T^i \cup T^e \) using one of the versions of the DPLL(\( T \)) procedure presented in the class. For theory reasoning in \( T \) use the Nelson-Oppen procedure.