Exercise 5.1: (3 P)
Let $\Sigma = \{0, s, +\}$. Consider the following formulae in the signature $\Sigma$:

1. $F_1 = \forall x \ (x + 0 \approx x)$
2. $F_2 = \forall x, y \ (x + s(y) \approx s(x + y))$
3. $F_3 = \forall x, y \ (x + y \approx y + x)$.

Find a $\Sigma$-structure in which $F_1$ and $F_2$ are valid but $F_3$ is not.

Exercise 5.2: (2 P)
Compute a clausal normal form for the following formula:

$\exists x \forall y (\forall z \ (p(y, z) \lor \neg x \approx y) \rightarrow (\forall z \ q(y, z) \land \neg r(x, y)))$

Exercise 5.3: (4 P)
Let $\Sigma = (\Omega, \Pi)$ with $\Omega = \{b/0, f/1\}$ and $\Pi = \{p/1\}$.

(a) Which is the universe of the Herbrand interpretations over this signature?

If $\mathcal{A}$ is a Herbrand interpretation over $\Sigma$ how are $b_\mathcal{A}$ and $f_\mathcal{A}$ defined?

(b) How many different Herbrand interpretations over $\Sigma$ do exist? Explain briefly.

(c) How many different Herbrand models over $\Sigma$ does the formula:

$p(f(f(b))) \land \forall x (p(x) \rightarrow p(f(x)))$  \hspace{1cm} (1)

have? Explain briefly.

(d) Every Herbrand model over $\Sigma$ of (1) is also a model of

$\forall x \ p(f(f(x)))$  \hspace{1cm} (2)

Give an example of an algebra that is a model of (1) but not of (2).
Exercise 5.4: (1 P)
Which of the following formulae is in the Bernays-Schönfinkel class?

(1) $\exists y \forall x \exists z ((p(x) \lor q(y)) \land (p(z) \lor \neg q(y))$
(2) $\forall x \exists y \forall z \exists u ((p(x) \lor q(y)) \land (q(y) \lor r(u, x))$
(3) $\exists y \exists z \forall x [(p(x) \lor q(y)) \land q(z)]$

Supplementary exercise

Exercise 5.5: (4 P)
Let $\Sigma = (\Omega, \Pi)$ be a signature and $X$ a set of variables. Let $\mathcal{A}$ be a $\Sigma$-structure and $\beta : X \to U_\mathcal{A}$ a variable assignment.

(1) Prove that for every formula $F \in F_\Sigma(X)$ and every $x \in X$, the truth values $A(\beta)(\forall x F)$ and $A(\beta)(\exists x F)$ do not depend on $\beta(x)$.

(2) Use (1) to show that if $G$ is a closed formula in $F_\Sigma(X)$, then the truth value of $G$ in $\mathcal{A}$ w.r.t. $\beta$, $A(\beta)(G)$ does not depend on the way $\beta$ is defined.

(3) Use (2) to prove that $\text{Th}(\text{Mod}(\mathcal{F})) = \{ G \in F_\Sigma(X) \text{ closed} \mid F \models G \}$.

Please submit your solution until Tuesday, December 7, 2021 at 12:00. Joint solutions prepared by up to three persons are allowed. Please do not forget to write your name on your solution.

Submission possibilities:

- In directory Homework 05 in OLAT
- By e-mail to sofronie@uni-koblenz.de with the keyword “Homework DP” in the subject.