Decision Procedures for Verification

First-Order Logic (2)

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Conventions

In what follows we will use the following conventions:

**constants** (0-ary function symbols) are denoted with $a, b, c, d, ...$

**function symbols** with arity $\geq 1$ are denoted
- $f, g, h, ...$ if the formulae are interpreted into arbitrary algebras
- $+, -, s, ...$ if the intended interpretation is into numerical domains

**predicate symbols** with arity 0 are denoted $P, Q, R, S, ...$

**predicate symbols** with arity $\geq 1$ are denoted
- $p, q, r, ...$ if the formulae are interpreted into arbitrary algebras
- $\leq, \geq, <, >$ if the intended interpretation is into numerical domains

**variables** are denoted $x, y, z, ...$
Until now:

**Syntax** (one-sorted signatures vs. many-sorted signatures)

**Semantics** (one-sorted signatures vs. many-sorted signatures)

Validity, Satisfiability, Entailment and Equivalence

**Algorithmic Problems; Decidability/Undecidability**
Goals

Identify:

• decidable fragments of first-order logic
• fragments of FOL for which satisfiability checking is easy
• Other decidable logical theories and theory fragments
Today

- Logical theories - Definition, Examples
- Normal forms $\leftrightarrow$ Resolution for first-order logic
Today

- Logical theories - Definition, Examples
- Normal forms $\iff$ Resolution for first-order logic
Let $\mathcal{A} \in \Sigma\text{-alg}$. The (first-order) theory of $\mathcal{A}$ is defined as

$$Th(\mathcal{A}) = \{ G \in F_{\Sigma}(X) \mid \mathcal{A} \models G \}$$

Problem of axiomatizability:

For which structures $\mathcal{A}$ can one axiomatize $Th(\mathcal{A})$, that is, can one write down a formula $F$ (or a recursively enumerable set $F$ of formulas) such that

$$Th(\mathcal{A}) = \{ G \mid F \models G \}?$$

Analogously for sets of structures.
Two Interesting Theories

Let $\Sigma_{\text{Pres}} = (\{0/0, s/1, +/2\}, \emptyset)$ and $\mathbb{Z}_+ = (\mathbb{Z}, 0, s, +)$ its standard interpretation on the integers.

$Th(\mathbb{Z}_+)$ is called **Presburger arithmetic** (M. Presburger, 1929).

(There is no essential difference when one, instead of $\mathbb{Z}$, considers the natural numbers $\mathbb{N}$ as standard interpretation.)

Presburger arithmetic is decidable in $3\text{EXPTIME}$ (D. Oppen, JCSS, 16(3):323–332, 1978), and in $2\text{EXPSPACE}$, using automata-theoretic methods (and there is a constant $c \geq 0$ such that $Th(\mathbb{Z}_+) \not\in \text{NTIME}(2^{2^{cn}})$).
Two Interesting Theories

However, \( \mathbb{N}_* = (\mathbb{N}, 0, s, +, *) \), the standard interpretation of \( \Sigma_{PA} = (\{0/0, s/1, +/2, */2\}, \emptyset) \), has as theory the so-called Peano arithmetic which is undecidable, not even recursively enumerable.

*Note:* The choice of signature can make a big difference with regard to the computational complexity of theories.
Logical theories

Syntactic view

first-order theory: given by a set $\mathcal{F}$ of (closed) first-order $\Sigma$-formulae.
the models of $\mathcal{F}$: $\text{Mod}(\mathcal{F}) = \{ \mathcal{A} \in \Sigma\text{-alg} \mid \mathcal{A} \models G, \text{ for all } G \text{ in } \mathcal{F} \}$

Semantic view

given a class $\mathcal{M}$ of $\Sigma$-algebras
the first-order theory of $\mathcal{M}$: $\text{Th}(\mathcal{M}) = \{ G \in F_\Sigma(X) \text{ closed} \mid \mathcal{M} \models G \}$
Theories

\[ \mathcal{F} \text{ set of (closed) first-order formulae} \]

\[ \text{Mod}(\mathcal{F}) = \{ A \in \Sigma\text{-alg} \mid A \models G, \text{ for all } G \in \mathcal{F} \} \]

\[ \mathcal{M} \text{ class of } \Sigma\text{-algebras} \]

\[ \text{Th}(\mathcal{M}) = \{ G \in F_\Sigma(X) \text{ closed} \mid \mathcal{M} \models G \} \]

\[ \text{Th}(\text{Mod}(\mathcal{F})) \text{ the set of formulae true in all models of } \mathcal{F} \]

represents exactly the set of consequences of \( \mathcal{F} \)
Theories

\( \mathcal{F} \) set of (closed) first-order formulae

\[
\text{Mod}(\mathcal{F}) = \{ A \in \Sigma\text{-alg} \mid A \models G, \text{ for all } G \text{ in } \mathcal{F} \}
\]

\( \mathcal{M} \) class of \( \Sigma \)-algebras

\[
\text{Th}(\mathcal{M}) = \{ G \in F_\Sigma(X) \text{ closed} \mid \mathcal{M} \models G \}
\]

\( \text{Th}(\text{Mod}(\mathcal{F})) \) the set of formulae true in all models of \( \mathcal{F} \)

represents exactly the set of consequences of \( \mathcal{F} \)

**Note:** \( \mathcal{F} \subseteq \text{Th}(\text{Mod}(\mathcal{F})) \) (typically strict)

\( \mathcal{M} \subseteq \text{Mod}(\text{Th}(\mathcal{M})) \) (typically strict)
Examples

1. Groups

Let $\Sigma = (\{e/0, */2, i/1\}, \emptyset)$

Let $F$ consist of all (universally quantified) group axioms:

\[
\begin{align*}
\forall x, y, z & \quad x \ast (y \ast z) \approx (x \ast y) \ast z \\
\forall x & \quad x \ast i(x) \approx e \quad \land \quad i(x) \ast x \approx e \\
\forall x & \quad x \ast e \approx x \quad \land \quad e \ast x \approx x
\end{align*}
\]

Every group $G = (G, e_G, \ast_G, i_G)$ is a model of $F$

$\text{Mod}(F)$ is the class of all groups

$F \subset \text{Th(}\text{Mod}(F))$
Examples

2. Linear (positive)integer arithmetic

Let $\Sigma = \{0/0, s/1, +/2\}, \{\leq /2\}$

Let $\mathbb{Z}^+ = (\mathbb{Z}, 0, s, +, \leq)$ the standard interpretation of integers.

$\{\mathbb{Z}^+\} \subset \text{Mod}(\text{Th}(\mathbb{Z}^+))$

3. Uninterpreted function symbols

Let $\Sigma = (\Omega, \Pi)$ be arbitrary

Let $\mathcal{M} = \Sigma\text{-alg}$ be the class of all $\Sigma$-structures

The theory of uninterpreted function symbols is $\text{Th}(\Sigma\text{-alg})$ the family of all first-order formulae which are true in all $\Sigma$-algebras.
4. Lists

Let $\Sigma = (\{\text{car}/1, \text{cdr}/1, \text{cons}/2\}, \emptyset)$

Let $\mathcal{F}$ be the following set of list axioms:

\[
\begin{align*}
\text{car}(\text{cons}(x, y)) & \approx x \\
\text{cdr}(\text{cons}(x, y)) & \approx y \\
\text{cons}(\text{car}(x), \text{cdr}(x)) & \approx x
\end{align*}
\]

$\text{Mod}(\mathcal{F})$ class of all models of $\mathcal{F}$

$\text{Th}_{\text{Lists}} = \text{Th}(\text{Mod}(\mathcal{F}))$ theory of lists (axiomatized by $\mathcal{F}$)
Goals

Identify:

• decidable theories; decidable fragments of first-order logic
• fragments of FOL for which satisfiability checking is easy

Methods:

• Adjust automated reasoning techniques
  (e.g. to obtaining efficient decision procedures)

  Extend methods for automated reasoning in propositional logic?
  Instantiation/reduction to propositional logic
  Extend the resolution calculus for first-order logic
Today

- Logical theories - Definition, Examples
- Normal forms $\iff$ Resolution for first-order logic
2.5 Normal Forms and Skolemization

Study of normal forms motivated by

- reduction of logical concepts,
- efficient data structures for theorem proving.

The main problem in first-order logic is the treatment of quantifiers. The subsequent normal form transformations are intended to eliminate many of them.
Prenex Normal Form

Prenex formulas have the form

\[ Q_1 x_1 \ldots Q_n x_n \ F, \]

where \( F \) is quantifier-free and \( Q_i \in \{\forall, \exists\} \); we call \( Q_1 x_1 \ldots Q_n x_n \) the quantifier prefix and \( F \) the matrix of the formula.
Prenex Normal Form

Computing prenex normal form by the rewrite relation $\Rightarrow_P$:

\[(F \leftrightarrow G) \Rightarrow_P (F \rightarrow G) \land (G \rightarrow F)\]

\[\neg QxF \Rightarrow_P \overline{Q}x\neg F \quad (\neg Q)\]

\[(QxF \rho G) \Rightarrow_P Qy(F[y/x] \rho G), \text{ y fresh, } \rho \in \{\land, \lor\}\]

\[(QxF \rightarrow G) \Rightarrow_P \overline{Q}y(F[y/x] \rightarrow G), \text{ y fresh}\]

\[(F \rho QxG) \Rightarrow_P Qy(F \rho G[y/x]), \text{ y fresh, } \rho \in \{\land, \lor, \rightarrow\}\]

Here $\overline{Q}$ denotes the quantifier dual to $Q$, i.e., $\overline{\forall} = \exists$ and $\overline{\exists} = \forall$. 
Example

\[ F := (\forall x((p(x) \lor q(x, y)) \land \exists z \ r(x, y, z))) \rightarrow ((p(z) \land q(x, z)) \land \forall z \ r(z, x, y)) \]
Example

\[ F := (\forall x ((p(x) \lor q(x, y)) \land \exists z r(x, y, z))) \rightarrow ((p(z) \land q(x, z)) \land \forall z r(z, x, y)) \]

\[ \Rightarrow_p \exists x' ((p(x') \lor q(x', y)) \land \exists z r(x', y, z)) \rightarrow ((p(z) \land q(x, z)) \land \forall z r(z, x, y)) \]
Example

\[ F := (\forall x ((p(x) \lor q(x, y)) \land \exists z r(x, y, z))) \rightarrow ((p(z) \land q(x, z)) \land \forall z r(z, x, y)) \]

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\[ \Rightarrow_p \exists x'(\exists z'((p(x') \lor q(x', y)) \land r(x', y, z'))) \rightarrow ((p(z) \land q(x, z)) \land \forall z r(z, x, y)) \]
Example

\[ F := (\forall x ((p(x) \lor q(x, y)) \land \exists z r(x, y, z))) \rightarrow ((p(z) \land q(x, z)) \land \forall z r(z, x, y)) \]

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\[ \Rightarrow_p \exists x' \exists z'((p(x') \lor q(x', y)) \land r(x', y, z'))) \rightarrow ((p(z) \land q(x, z)) \land \forall z r(z, x, y)) \]

\[ \Rightarrow_p \exists x' \forall z'(((p(x') \lor q(x', y)) \land r(x', y, z'))) \rightarrow ((p(z) \land q(x, z)) \land \forall z r(z, x, y)) \]
Example

\[ F := (\forall x ((p(x) \lor q(x, y)) \land \exists z r(x, y, z))) \rightarrow ((p(z) \land q(x, z)) \land \forall z r(z, x, y)) \]

\[ \Rightarrow_p \exists x' ((p(x') \lor q(x', y)) \land \exists z r(x', y, z))) \rightarrow ((p(z) \land q(x, z)) \land \forall z r(z, x, y)) \]

\[ \Rightarrow_p \exists x' (\exists z' ((p(x') \lor q(x', y)) \land r(x', y, z'))) \rightarrow ((p(z) \land q(x, z)) \land \forall z r(z, x, y)) \]

\[ \Rightarrow_p \exists x' \forall z' ((p(x') \lor q(x', y)) \land r(x', y, z')) \rightarrow ((p(z) \land q(x, z)) \land \forall z r(z, x, y)) \]

\[ \Rightarrow_p \exists x' \forall z' ((p(x') \lor q(x', y)) \land r(x', y, z')) \rightarrow \forall z'' ((p(z) \land q(x, z)) \land r(z'', x, y)) \]
Example

\[ F := (\forall x ((p(x) \lor q(x, y)) \land \exists z \ r(x, y, z))) \rightarrow ((p(z) \land q(x, z)) \land \forall z \ r(z, x, y)) \]

\[
\Rightarrow_p \ \exists x'( (p(x') \lor q(x', y)) \land \exists z \ r(x', y, z)) \rightarrow ((p(z) \land q(x, z)) \land \forall z \ r(z, x, y)) \\
\Rightarrow_p \ \exists x' (\exists z' ( (p(x') \lor q(x', y)) \land r(x', y, z'))) \rightarrow ((p(z) \land q(x, z)) \land \forall z \ r(z, x, y)) \\
\Rightarrow_p \ \exists x' \forall z' ( (p(x') \lor q(x', y)) \land r(x', y, z')) \rightarrow ( (p(z) \land q(x, z)) \land \forall z \ r(z, x, y)) \\
\Rightarrow_p \ \exists x' \forall z' ( (p(x') \lor q(x', y)) \land r(x', y, z')) \rightarrow \forall z'' ( (p(z) \land q(x, z)) \land r(z'', x, y)) \\
\Rightarrow_p \ \exists x' \forall z' \forall z'' ( (p(x') \lor q(x', y)) \land r(x', y, z')) \rightarrow ( (p(z) \land q(x, z)) \land r(z'', x, y)) \]
Skolemization

**Intuition:** remove $\exists y$.

For this:

- we introduce a concrete choice function $sk_y$ computing $y$ from all the arguments $y$ depends on (i.e. from all variables $x_1, \ldots, x_n$ which occur universally quantified before $y$ in the quantifier prefix)
- We replace $y$ with $sk_y(x_1, \ldots, x_n)$ everywhere in the scope of $\exists y$.

Transformation $\Rightarrow_S$ (to be applied outermost, not in subformulas):

$$\forall x_1, \ldots, x_n \exists y F \Rightarrow_S \forall x_1, \ldots, x_n F[sk_y(x_1, \ldots, x_n)/y]$$

where $sk_y/n$ is a new function symbol (Skolem function).
Skolemization

Together: $F \Rightarrow_P^{\text{prenex}} G \Rightarrow_S^{\text{prenex}} H \Rightarrow S$, no $\exists$

Theorem 2.9:

Let $F$, $G$, and $H$ as defined above and closed. Then

(i) $F$ and $G$ are equivalent.

(ii) $H \models G$ but the converse is not true in general.

(iii) $G$ satisfiable (wrt. $\Sigma$-alg) $\iff$ $H$ satisfiable (wrt. $\Sigma'$-Alg)

where $\Sigma' = (\Omega \cup SKF, \Pi)$, if $\Sigma = (\Omega, \Pi)$. 

28
Clausal Normal Form (Conjunctive Normal Form)

\[
(F \leftrightarrow G) \implies_K (F \to G) \land (G \to F) \\
(F \to G) \implies_K (\neg F \lor G) \\
(\neg (F \lor G)) \implies_K (\neg F \land \neg G) \\
(\neg (F \land G)) \implies_K (\neg F \lor \neg G) \\
\neg \neg F \implies_K F \\
(F \land G) \lor H \implies_K (F \lor H) \land (G \lor H) \\
(F \land \top) \implies_K F \\
(F \land \bot) \implies_K \bot \\
(F \lor \top) \implies_K \top \\
(F \lor \bot) \implies_K F
\]

These rules are to be applied modulo associativity and commutativity of \( \land \) and \( \lor \). The first five rules, plus the rule \((\neg Q)\), compute the negation normal form (NNF) of a formula.
The Complete Picture

\[ F \Rightarrow^*_P Q_1 y_1 \ldots Q_n y_n \ G \]  
\[ \Rightarrow^*_S \forall x_1, \ldots, x_m \ H \]  
\[ \Rightarrow^*_K \forall x_1, \ldots, x_m \ \left( \bigwedge_{i=1}^{k} \bigvee_{j=1}^{n_i} L_{ij} \right) \]  
leave out \[ F' \]

\[ N = \{ C_1, \ldots, C_k \} \] is called the clausal (normal) form (CNF) of \( F \).

*Note:* the variables in the clauses are implicitly universally quantified.

**Theorem 2.10:**
Let \( F \) be closed. Then \( F' \models F \). (The converse is not true in general.)

**Theorem 2.11:**
Let \( F \) be closed. Then \( F \) is satisfiable iff \( F' \) is satisfiable iff \( N \) is satisfiable.
Example

Given:  \( \exists u \forall w (\exists x (p(w, x, u) \lor \forall y (q(w, x, y) \land \exists z r(y, z))) ) \)
Example

**Given:** \( \exists u \forall w (\exists x (p(w, x, u) \lor \forall y (q(w, x, y) \land \exists z r(y, z))) ) \)

**Prenex Normal Form:**

\[ \Rightarrow^*_p \exists u \forall w \exists x \forall y \exists z ((p(w, x, u) \lor (q(w, x, y) \land r(y, z)))) \]
Example

**Given:** \[ \exists u \forall w (\exists x (p(w, x, u) \lor \forall y (q(w, x, y) \land \exists z r(y, z)))) \]

**Prenex Normal Form:**

\[ \Rightarrow^p \exists u \forall w \exists x \forall y \exists z ((p(w, x, u) \lor (q(w, x, y) \land r(y, z)))) \]

**Skolemisation:**

\[ \Rightarrow^s \forall w \forall y ((p(w, \text{sk}_x(w)), \text{sk}_u) \lor (q(w, \text{sk}_x(w), y) \land r(y, \text{sk}_z(w, y)))) \]
Example

Given:
\[ \exists u \forall w(\exists x (p(w, x, u) \lor \forall y (q(w, x, y) \land \exists z r(y, z)))) \]

Prenex Normal Form:
\[ \Rightarrow^*_p \exists u \forall w \exists x \forall y \exists z ((p(w, x, u) \lor (q(w, x, y) \land r(y, z)))) \]

Skolemisation:
\[ \Rightarrow^*_s \forall w \forall y((p(w, sk_x(w), sk_u) \lor (q(w, sk_x(w), y) \land r(y, sk_z(w, y)))) \]

Clause normal form:
\[ \Rightarrow^*_k \forall w \forall y [(p(w, sk_x(w), sk_u) \lor q(w, sk_x(w), y)) \land
(p(w, sk_x(w), sk_u) \lor r(y, sk_y(w, y)))] \]

Set of clauses:
\[ \{ p(w, sk_x(w), sk_u) \lor q(w, sk_x(w), y), \quad p(w, sk_x(w), sk_u) \lor r(y, sk_y(w, y)) \} \]
Optimization

Here is lots of room for optimization since we only can preserve satisfiability anyway:

- size of the CNF exponential when done naively;
- want to preserve the original formula structure;
- want small arity of Skolem functions.
2.6 Herbrand Interpretations

From now an we shall consider PL without equality. \( \Omega \) shall contains at least one constant symbol.

A **Herbrand interpretation** (over \( \Sigma \)) is a \( \Sigma \)-algebra \( \mathcal{A} \) such that

- \( U_\mathcal{A} = T_\Sigma \) (\( = \) the set of ground terms over \( \Sigma \))
- \( f_\mathcal{A} : (s_1, \ldots, s_n) \mapsto f(s_1, \ldots, s_n), f/n \in \Omega \)

\[
f_\mathcal{A}(\triangle, \ldots, \triangle) = \begin{array}{c} f \\ \triangle \\ \ldots \\ \triangle \end{array}
\]
Herbrand Interpretations

In other words, values are fixed to be ground terms and functions are fixed to be the term constructors. Only predicate symbols $p/m \in \Pi$ may be freely interpreted as relations $p_{\mathcal{A}} \subseteq T^m_{\Sigma}$.

**Proposition 2.12**

Every set of ground atoms $I$ uniquely determines a Herbrand interpretation $\mathcal{A}$ via

$$(s_1, \ldots, s_n) \in p_{\mathcal{A}} \iff p(s_1, \ldots, s_n) \in I$$

Thus we shall identify Herbrand interpretations (over $\Sigma$) with sets of $\Sigma$-ground atoms.
**Herbrand Interpretations**

*Example:* \( \Sigma_{Pres} = (\{0/0, s/1, +/2\}, \{</2, \leq/2\}) \)

\( \mathbb{N} \) as Herbrand interpretation over \( \Sigma_{Pres} \):

\[ I = \{ 0 \leq 0, 0 \leq s(0), 0 \leq s(s(0)), \ldots, \\
0 + 0 \leq 0, 0 + 0 \leq s(0), \ldots, \\
\ldots, (s(0) + 0) + s(0) \leq s(0) + (s(0) + s(0)) \\
\ldots, s(0) + 0 < s(0) + 0 + 0 + s(0) \\
\ldots \} \]
Existence of Herbrand Models

A Herbrand interpretation $I$ is called a Herbrand model of $F$, if $I \models F$.

**Theorem 2.13**

Let $N$ be a set of $\Sigma$-clauses.

$N$ satisfiable $\iff$ $N$ has a Herbrand model (over $\Sigma$)

$\iff$ $G_\Sigma(N)$ has a Herbrand model (over $\Sigma$)

where $G_\Sigma(N) = \{ C\sigma \text{ ground clause} \mid C \in N, \sigma : X \rightarrow T_\Sigma \}$ is the set of ground instances of $N$.

(Proof – completeness proof of resolution for first-order logic.)
Example of a $G_\Sigma$

For $\Sigma_{Pres}$ one obtains for

$$C = (x < y) \lor (y \leq s(x))$$

the following ground instances:

$(0 < 0) \lor (0 \leq s(0))$

$(s(0) < 0) \lor (0 \leq s(s(0)))$

$\ldots$

$(s(0) + s(0) < s(0) + 0) \lor (s(0) + 0 \leq s(s(0) + s(0)))$

$\ldots$
Consequences of Herbrans’s theorem

Decidability results.

- Formulae without function symbols and without equality

  The Bernays-Schönfinkel Class $\exists^* \forall^*$
The Bernays-Schönfinkel Class

\[ \Sigma = (\Omega, \Pi) \], \ \Omega \text{ is a finite set of constants} \]

The Bernays-Schönfinkel class consists only of sentences of the form

\[ \exists x_1 \ldots \exists x_n \forall y_1 \ldots \forall y_m F(x_1, \ldots, x_n, y_1, \ldots, y_n) \]
The Bernays-Schönfinkel Class

Σ = (Ω, Π), Ω is a finite set of constants

The Bernays-Schönfinkel class consists only of sentences of the form

$$\exists x_1 \ldots \exists x_n \forall y_1 \ldots \forall y_m F(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

Idea: CNF translation:

$$\exists x_1 \forall y_1 F_1 \land \cdots \exists x_n \forall y_n F_n$$

$$\Rightarrow_P \exists x_1 \ldots \exists x_n \forall y_1 \ldots \forall y_n F (x_1, \ldots, x_n, y_1, \ldots, y_n)$$

$$\Rightarrow_S \forall y_1 \ldots \forall y_m F (c_1, \ldots, c_n, y_1, \ldots, y_n)$$

$$\Rightarrow_K \forall y_1 \ldots \forall y_m \land \lor L_i ((c_1, \ldots, c_n, y_1, \ldots, y_n)$$

$$\overline{c}_1, \ldots, \overline{c}_n$$ are tuples of Skolem constants
The Bernays-Schönfinkel Class

\[ \Sigma = (\Omega, \Pi), \, \Omega \text{ is a finite set of constants} \]

The Bernays-Schönfinkel class consists only of sentences of the form

\[ \exists x_1 \ldots \exists x_n \forall y_1 \ldots \forall y_m F(x_1, \ldots, x_n, y_1, \ldots, y_n) \]

**Idea:** CNF translation:

\[ \exists x_1 \forall y_1 F_1 \land \ldots \exists x_n \forall y_n F_n \]

\[ \Rightarrow^*_K \forall y_1 \ldots \forall y_m \land \lor L_i((\overline{c}_1, \ldots, \overline{c}_n, \overline{y}_1, \ldots, \overline{y}_n)) \]

\[ \overline{c}_1, \ldots, \overline{c}_n \text{ are tuples of Skolem constants} \]

The Herbrand Universe is finite \( \Rightarrow \) decidability