Exercise 9.2  Prove the following equivalences of CTL formulae:

1. $\neg E O F \equiv ADT F$

   $\neg E O F$ is an abbreviation for the formula $\neg E O T G$.

   Therefore, $ADT F \equiv \neg E O T F \equiv \neg E O F$.

2. $E (F U G) \equiv G V (F \land E O E (F U G))$

   We show that in every transition system $T = (S, \rightarrow, L)$ and every $s \in S$,
   $(T, s) \models E (F U G)$ if and only if $(T, s) \models G V (F \land E O E (F U G))$:

   Proof. Let $T$ be a transition system and $s$ be a state of $T$.

   $(T, s) \models E (F U G)$ if and only if there exists a computation $\pi = s_0 \rightarrow s_1 \rightarrow \cdots$ where $s_0 = s$ such that

   $\exists \mu \geq 0$ with $(T, s_0) \models G$ and
   $\forall k \in 0, \ldots, \mu - 1 : (T, s_k) \models F$.

   Therefore, $E (F U G) \equiv G V (F \land E O E (F U G))$ if and only if

   $(T, s) \models G$ or $(T, s) \models F$ and $(T, s) \models E O E (F U G)$

   if and only if $(T, s) \models G$ or $(T, s) \models F$ and there exists
   $s_i \in S \cup s \downarrow s_i$ such that $(T, s_i) \models E (F U G)$

   if and only if $(T, s) \models G$ or $(T, s) \models F$ and there exists
   $s_i \in S \cup s \downarrow s_i$ such that there exists
   $T = s_i \rightarrow s_{i+1} \rightarrow \cdots$ for which
   $\exists \mu \geq 1 : (T, s_0) \models G$ and
   $\forall k \in 1, \ldots, \mu - 1 : (T, s_k) \models F$.

   It is easy to check that $(\forall)$ and $(\exists)$ are equivalent.
(3) \[ \text{EOF} = F \land \text{EOF} \]

Proof: We show that for every transition system \( T \) and every state \( s \) of \( T \),
\[ (T, s) \not\in \text{EOF} \quad \text{iff} \quad (T, s) \not\in F \land \text{EOEOF}. \]

Let \( T \) be a transition system and \( s \) be a state of \( T \).

\[ (T, s) \not\in \text{EOF} \quad \text{iff} \quad \text{there exists a computation } T = S_0 \rightarrow S_1 \rightarrow \ldots \text{ with } S_0 = s \]
\[ \text{such that } \#u > 0 : (T, Sw) = F. \]

\[ (T, s) \not\in F \land \text{EOEOF} \quad \text{iff} \quad \exists (T, s) = F \text{ and } \]
\[ (T, s) \not\in \text{EOF} \]
\[ \text{iff} \quad (T, s) = F \text{ and } \]
\[ \text{there exists } S_1 \text{ with } s \rightarrow S_1 \text{ such that } (T, S_1) = \text{EOF}. \]
\[ \text{iff} \quad (T, s) = FF \text{ and } \]
\[ \text{there exists } S_1 \text{ with } s \rightarrow S_1 \text{ and } \]
\[ \text{there exists a computation } T = S_1 \rightarrow S_2 \rightarrow \ldots \]
\[ \text{such that } \#u > 1 : (T, Sw) = F. \]
\[ \text{iff} \quad \text{there exists a computation } T = S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow \ldots \]
\[ \text{such that } (T, S_0) = (T, s) = FF \text{ and } \]
\[ \#u > 1 : (T, Sw) = F. \]
\[ \text{iff} \quad \text{there exists a computation } T = S_0 \rightarrow S_1 \rightarrow \ldots \text{ with } S_0 = s \]
\[ \text{such that } \#u > 0 : (T, Sw) = F. \]
\[ \text{iff} \quad (T, s) = \text{EOF}. \]
(4) \( TA(FUG) \equiv E(7G \cup (7F \setminus 7G)) \cup E07G .\)

**Proof:** We show that in every transition system \( T=(S, \rightarrow, L) \) and every \( S \in S \):

\( (T,S) \equiv TA(FUG) \) \iff \( (T,S) \equiv E(7G \cup (7F \setminus 7G)) \cup E07G .\)

Let \( T \) be a transition system and \( S \) be a state of \( T .\)

\( (T,S) \equiv TA(FUG) \) \iff

- It is not true that for all computations \( \Pi=S_0 \rightarrow S_1 \rightarrow \ldots \) with \( S_0=S \) there is an \( n \geq 0 \) such that \( n \in \mathbb{Z} \) and \( T(S_0) = FG \) and \( T(S_n) = FG \)

  \iff there exists a computation \( \Pi=S_0 \rightarrow S_1 \rightarrow \ldots \) with \( S_0=S \) such that for all \( n \geq 0 \) \( (T(S_n)) = FG \) or \( \exists k \in \{0, \ldots, n \} : (T,S) = TF \)

  \iff there exists a computation \( \Pi=S_0 \rightarrow S_1 \rightarrow \ldots \) with \( S_0=S \) such that either for all \( n \geq 0 \) \( (T(S_n)) = FG \) or there exists \( n_0 \geq 0 \) such that \( (T(S_n)) = FG \) for all \( n \geq n_0 \) and \( \exists k \in \{0, \ldots, n_0 \} : (T,S) = TF \)

- Case distinction made clear.

  - If \( S \) follows with \( (T(S_n)) = FG \) we choose the smallest such \( n_0 \)

  - Then (1) holds.

\( \exists (A \lor B) \equiv \exists A \lor \exists B \)

\[ (A \lor B) \equiv (A \land B) \]

\[ (A \land B) \equiv (A \lor B) \]

4. There exists a computation \( \Pi=S_0 \rightarrow S_1 \rightarrow \ldots \) with \( S_0=S \) such that for all \( n \geq 0 \) \( (T(S_n)) = FG \) or there exists \( n_0 \geq 0 \) with \( (T(S_n)) = FG \) for all \( n \geq n_0 \) and \( \exists k \in \{0, \ldots, n_0 \} : (T,S) = TF .\)
\((T,S) = E(7G \cup (7F \land 7G)) \lor E(7G)\) 

iff \((T,S) = E(7G \cup (7F \land 7G))\) or \((T,S) = E(7G)\).

iff there exists a computation \(T = S_0 \rightarrow S_1 \rightarrow \cdots \) with \(S_0 = S\) such that \(\forall u \geq 0\) \((T, Su) \neq 7G\),

or

there exists a computation \(T = S_0 \rightarrow S_1 \rightarrow \cdots\) with \(S_0 = S\) such that \(\exists n_0 \geq 0\) such that \((T, S_{n_0}) = 7F \land 7G\),

and \(\forall k \leq n_0\) \((T, S_k) = 7G\).

We now show that \(\oplus \iff \ominus\).

\(\oplus \Rightarrow \ominus\) \(\Box\) If there exists a computation \(T = S_0 \rightarrow S_1 \rightarrow \cdots\) with \(S_0 = S\) such that \(\forall u \geq 0\) \((T, Su) = 7G\), then \(\ominus\) holds.

Assume now that there exists a computation \(T = S_0 \rightarrow S_1 \rightarrow \cdots\) with \(S_0 = S\) such that \(\exists n_0\) with \((T, S_{n_0}) \neq 7G\), \((T, Su) \neq 7G\) for all \(u < n_0\), and \(\exists k \leq n_0 - 1\) such that \((T, S_k) = 7F\).

We choose \(n_0\) to be the \(k \leq n_0 - 1\) for which \((T, S_k) = 7F\).

Then \((T, S_{n_0}) = 7G\) because \(k \leq n_0\), \(n_0 \neq 7F \land 7G\).

In addition, we know that for all \(k \leq n_0 \leq n_0 - 1\), we have \((T, S_k) = 7G\). Thus \(\ominus\) holds also in this case.

\(\ominus \Rightarrow \oplus\) \(\Box\) If there exists a computation \(T = S_0 \rightarrow S_1 \rightarrow \cdots\) with \(S_0 = S\) such that \(\forall u \geq 0\) \((T, Su) \neq 7G\), then \(\oplus\) holds.

This is not the case and \(\ominus\).

Assume now that there exists a computation \(T = S_0 \rightarrow S_1 \rightarrow \cdots\) with \(S_0 = S\) such that \(\exists n_0\) such that \((T, S_{n_0}) \neq 7F \land 7G\) and \(\forall k \leq n_0\) \((T, S_k) = 7G\).

Then \(\exists n_0\) such that \((T, S_{n_0}) \neq 7F\), and \(\forall u < n_0\) \((T, Su) = 7G\).

\(\exists k \leq n_0\ \exists u \leq n_{0-1}\) such that \((T, S_k) = 7F\).

Thus \(\ominus\) holds also in this case.