\[ \Sigma = (\Sigma, \Pi) \text{ where } \Pi = \emptyset \text{ (the only predicate symbol is equality) } \]

- The universality property of free algebras

Let \( K \) be a class of \( \Sigma \)-algebras and \( n \) be the congruence relation on \( T_{\Sigma}(x) \) defined by \( t_1 \equiv t_2 \) if \( K \models t_1 = t_2 \) (i.e. if \( a \models t_1 = t_2 \) for all \( a \in K \)).

Let \( T_{\Sigma}(x) / \sim \) be the algebra defined by:

\[
\overline{T_{\Sigma}(x)} / \sim = ( T_{\Sigma}(x) / \sim , \{ f : \overline{T_{\Sigma}(x)} / \sim \to \overline{T_{\Sigma}(x)} / \sim \mid f \in \Sigma \} )
\]

where \( \overline{T_{\Sigma}(x)} / \sim = \{ [t] \mid t \in T_{\Sigma}(x) \} \) and \( [t] = \{ t' \in T_{\Sigma}(x) \mid t \equiv t' \} \).

- If \( f/t \in \Sigma \) then
  \[
  f_{T_{\Sigma}(x) / \sim} ([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]
  \]

  (The definition does not depend on the representatives because if \( [t_1] = [t'_1], \ldots, [t_n] = [t'_n] \)
  then \( t_1 \equiv t'_1, \ldots, t_n \equiv t'_n \).

  Hence, as \( \sim \) is a congruence relation,

\[
 f(t_1, \ldots, t_n) \equiv f(t'_1, \ldots, t'_n).
\]

**Definition:** Let \( A = (U_A, \{ f_A \mid f \in \Sigma \}) \) be \( \Sigma \)-algebras,

\( B = (U_B, \{ f_B \mid f \in \Sigma \}) \)

A homomorphism \( h : A \to B \) is a map

\[ h : U_A \to U_B \text{ with the property that for every } f \in \Sigma \text{ with } a(f) = n, \]

and for every \( a_1, \ldots, a_n \in U_A \)

\[ h( f_A(a_1, \ldots, a_n) ) = f_B( h(a_1), \ldots, h(a_n) ) \].
Theorem (the universal property of the free algebra): Let \( A \in \mathcal{K} \). Then for every \( \beta : X \to A \) there exists a unique homomorphism of \( \Sigma \)-algebras \( \beta' : T_\Sigma(x)/\sim \to A \) with the property that \( \beta'(\lfloor x \rfloor) = \beta(x) \) for all \( x \in X \).

Proof: Let \( \beta : X \to A \). We can extend (homomorphically) \( \beta \) to \( \beta' : T_\Sigma(x) \to A \), where \( A(\beta)(x) = x \),

we have \( x \xrightarrow{\beta} A(\beta) \)

Consider now the surjection \( p : T_\Sigma(x) \to T_\Sigma(x)/\sim \) defined by \( p(t) = [t] = \{ t' \in T_\Sigma(x) \mid t \sim t' \} \).

we define \( \beta' : T_\Sigma(x)/\sim \to A \) by \( \beta'([t]) = A(\beta)(t) \).

we have \( x \xrightarrow{\beta} A \)

\( \xrightarrow{\beta'} \)

we need to prove:

1. \( \beta' \) well-defined, i.e.
   
   if \( [t] = [t'] \) then \( \beta'(t) = \beta'(t') \)

2. \( \beta' \) homomorphism of \( \Sigma \)-algebras

3. \( \beta'([x]) = \beta(x) \) for all \( x \in X \)

4. If \( h : T_\Sigma(x)/\sim \to A \) is such that
   
   \( h([x]) = \beta(x) \) and \( h \) is a homomorphism of \( \Sigma \)-algebras
   
   then \( h = \beta' \).
1. $\beta'$ is well-defined.

Assume $[t] = [t']$,
then $t \equiv t'$, i.e. $[K] = t = t'$, i.e. $B \equiv t = t' \forall B \in K$.

Since $A \in K$ it follows that $A \equiv t = t'$,
hence for all $\overline{a} : x \rightarrow A$,$A(\overline{a})(t) = A(\overline{a})(t')$.

For $\overline{a} = \overline{b}$ we therefore have $U(\overline{a})(t) = U(\overline{a})(t')$.

Hence, $\beta'(\{t\}) = U(\overline{a})(t) = U(\overline{a})(t') = \beta'(\{t'\})$.

2. $\beta'$ is a homomorphism of $\Sigma$-algebras.

$\beta'(\sum_{i=1}^n f_i(x)) = \beta'(\sum_{i=1}^n f_i(x))$ = $A(\overline{a})(f(t_1), \ldots, f(t_n))$

Def. of $\beta'$

$\beta'(f(x)) = A(\overline{a})(f(t_1), \ldots, f(t_n))$

Def. of $A(\overline{a})$

$\beta'(\{x\}) = \beta(x)$ for all $x \in X$.

3. $\beta'(\{x\}) = \beta(x)$ for all $x \in X$

Def. of $\beta'$

$\beta'(\{x\}) = A(\overline{a})(x) = \beta(x)$.

4. $\beta'$ is the unique extension.

Let $h : T_\Sigma(x) \rightarrow A$ with $h([x]) = \beta(x) \in h \Sigma$-homomorphism.

We can prove by structural induction that for every $t \in T_\Sigma(x)$

$h([t]) = A(\overline{a})(t)$.

\[ \begin{array}{c}
\text{If } t = x: \quad h([x]) = \beta(x) = U(\overline{a})(x) \\
\text{If } t = \sum_{i=1}^n t_i \text{ and term for } t_i \in T_\Sigma(x) \Rightarrow t = U(t_1, \ldots, t_n) \text{ for } t_i \in T_\Sigma(x).
\end{array} \]

5. Assume property holds for $t_1, \ldots, t_n$.

Prove it holds for $t$ with $h$ homomorphism.