On the Spanning Ratio of Partial Delaunay Triangulation

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Abstract—Partial Delaunay triangulation (PDT) is a well-known subgraph construction that has already been used for years in the context of geographic routing and topology control. So far, it has been unknown if partial Delaunay triangulation is a network spanner. Network spanners are those subgraph constructions which maintain the length of the shortest path between any pair of nodes up to a constant factor. This factor is also referred to as the spanning ratio. In this work we prove that partial Delaunay triangulation is a network spanner for unit disk graphs. Furthermore, from our proof follows immediately that the spanning ratio of PDT is less than or equal to \( \frac{1 + \sqrt{5}}{4} \pi^2 \).

Index Terms—Wireless ad hoc and sensor network, spanner, partial Delaunay triangulation, localized algorithm

I. INTRODUCTION

Localized geographic routing algorithms in wireless multihop networks (see e.g., [1], [2]) perform message forwarding just by taking the current node’s, its neighbors’, and the destination node’s locations into account. Localized geographic routing provably guarantees message delivery if the utilized network graph is planar [3]. Typically the network graph just connecting the nodes which can directly communicate with each other is not planar. Thus, before localized geographic forwarding can take place, a forwarding node has to compute a local view on a planar subgraph of the network graph.

Several well understood localized planarization techniques can be applied: Gabriel graph (GG) [4], relative neighborhood graph (RNG) [5], restricted Delaunay graph (RDG) [6], and localized Delaunay triangulation (LDel) [7]. The advantage of GG and RNG over RDG and LDel is that each node has to know only its one hop neighbors to decide its incident edges in GG and in RNG. In contrast, the subgraphs RDG and LDel require neighborhood information to be exchanged along two hops. The advantage of RDG and LDel over GG and RNG is however that they have constant Euclidean stretch.

A subgraph has a constant Euclidean stretch if and only if the length of the shortest path connecting any two nodes in the subgraph is upper bounded by a constant times the length of the shortest path connecting the two nodes in the original network graph.

In 2004, Li et al. [8] proposed the partial Delaunay triangulation (PDT), an interesting planar subgraph construction variant, which can be computed locally using only one-hop neighborhood information. The resulting subgraph contains the GG and the RNG, and is contained in the unit Delaunay triangulation, which is known to have a constant stretch. Thus, regarding the spanning ratio, it lies in between these graph constructions. So far, it has remained an open question if its spanning ratio is constant or not. Some authors (e.g., [9], [10]) conjectured that PDT does not have constant spanning ratio, whereas others (e.g., [11], [12]) state this to be unknown.

Until 2011, it even remained an open question if localized planar spanner construction just with one-hop neighborhood information is possible at all. This question has recently been answered with the partial unit Delaunay graph (PuDel), described in the seminal work from Xu et al. [13]. PuDel is a one-hop localized planar subgraph construction, which has a constant spanning ratio of at most \( \frac{1 + \sqrt{5}}{4} \pi^2 \approx 7.98 \).

Now having answered the general question about one-hop localized planar spanner construction with one example graph construction, however, one important question remains: is the well-known PDT construction a spanner, and if yes, what is its spanning ratio?

In this paper, we show that although from an algorithmic point of view the constructions of PDT and PuDel are quite different, the resulting subgraphs are equivalent. Consequently, PDT has a constant spanning ratio of at most \( \frac{1 + \sqrt{5}}{4} \pi^2 \) as well.

Since we show the constant spanning ratio of PDT by showing equivalence between PuDel and PDT, the correctness of our result directly depends on the correctness of the spanning ratio analysis of PuDel.
As [13] makes use of non-conventional notation, leaves several proofs to the reader, and contains some subtle inaccuracies, their line of argumentation is not evident in its entirety. In order to assist the reader and to guarantee correctness of our result, we present in Section III a simplified, extended, and rigorous version of Xu et al.’s proof. Our main result, the equivalence of PuDel and PDT, is then presented in Section IV.

II. PRELIMINARIES

A. Network graph

We consider network graphs where each node is represented by a distinct point in the Euclidean plane. The Euclidean distance between any two such points, \( p \) and \( q \), is denoted by \( \|pq\| \). Moreover, the \( x \)- and \( y \)-components of \( p \) are denoted by \( x(p) \) and \( y(p) \), respectively.

Throughout this work, we consider a finite node set \( S \subset \mathbb{R}^2, 3 \leq |S| < \infty \). The nodes in \( S \) are connected by a unit disk graph \( \text{UDG}(S) \). Given a positive constant \( R \), such a graph connects all nodes \( u, v \in S \) with an edge iff \( \|uv\| \leq R \). We always assume that \( \text{UDG}(S) \) is a connected graph.

The one-hop neighborhood \( \text{N}(u) \) of a node \( u \) is the set of all nodes reachable from \( u \) in one hop, including \( u \) itself, i.e., \( \text{N}(u) = \{ v \in S : \|uv\| \leq 1 \} \).

B. Elementary geometric constructions

By \( \overline{uv} \) we denote the line segment between \( u \) and \( v \). By \( B(u, v) = \{ x \in \mathbb{R}^2 : \|ux\| = \|vx\| \} \) we denote the bisector of \( u \) and \( v \). \( B(u, v) \) divides the Euclidean plane into the closed half-plane \( H(u, v) = \{ x \in \mathbb{R}^2 : \|ux\| \leq \|vx\| \} \) containing node \( u \) and the closed half-plane \( H(v, u) = \{ x \in \mathbb{R}^2 : \|vx\| \leq \|vx\| \} \) containing node \( v \) (see Fig. 1a). As we are considering closed half-planes, note that \( H(u, v) \cap H(v, u) = B(u, v) \) holds.

For any point \( x \in \mathbb{R}^2 \), let \( C_r(x) \) denote the circle centered at \( x \) with radius \( r \). For convenience, we often use the notation \( C_r(x) \cap S \) to denote the subset \( \{ v \in S : \|vx\| \leq r \} \subseteq S \). By \( C(u, v, w) \) we denote the unique circle that has \( u, v, \) and \( w \) on its circle boundary.

The disk that has the line segment \( \overline{uv} \) as its diameter is denoted by \( \text{Disk}(u, v) \).

Finally, we let \( \angle uvw \) denote the internal angle of the intersecting line segments \( \overline{uv} \) and \( \overline{vw} \), where \( u, v, w \in S \).

C. Voronoi diagram and Delaunay triangulation

In the remainder of this section, we introduce the geometric concepts which are required for eventually defining PDT and PuDel, the subjects of our study. We begin with the well known concept of Voronoi diagrams.

**Definition 1 (Voronoi diagram [14]):** Let \( u \in S \). We call the region

\[
\text{VR}_S(u) = \bigcap_{v \in S \setminus \{u\}} H(u, v)
\]

the (ordinary) Voronoi region of node \( u \) w.r.t. the set \( S \). Furthermore, we call the set

\[
\text{VD}(S) = \{ \text{VR}_S(u_1), ..., \text{VR}_S(u_n) \},
\]

where \( \{ u_1, ..., u_n \} = S \), the (planar ordinary) Voronoi diagram, generated by \( S \) (see Fig. 1b).

As we are defining the Voronoi region as a closed set, the boundary of it may consist of line segments, half-lines or infinite lines, which we will call Voronoi edges. Moreover, if \( \text{VR}_S(u) \cap \text{VR}_S(v) \neq \emptyset \) for \( u, v \in S \), this intersection gives a Voronoi edge, which may be degenerated into a single point. The end points of Voronoi edges are referred to as Voronoi vertices.

When there exists at least one Voronoi vertex at which four or more Voronoi edges meet in the Voronoi diagram \( \text{VD}(S) \), we call \( \text{VD}(S) \) degenerated, otherwise it is said to be non-degenerated [14]. A Voronoi diagram is degenerated if four or more nodes in \( S \) are co-circular. As degenerated cases make proofs involving Voronoi-related concepts lengthy without adding new insights, we shall assume the following throughout the entire paper:

**Assumption 1 (Non-degeneracy [14]):** Every Voronoi vertex in a Voronoi diagram \( \text{VD}(S) \) has exactly three Voronoi edges. i.e., no four points in \( S \) are co-circular.

**Definition 2 (Delaunay triangulation [15]):** The Delaunay triangulation of the set \( S \), denoted by \( \text{Del}(S) \), is an undirected graph connecting each pair of nodes from \( S \), whose Voronoi regions share a Voronoi edge in the Voronoi diagram \( \text{VD}(S) \) (see Fig. 1b). The edges in \( \text{Del}(S) \) are referred to as Delaunay edges.

In order for the Delaunay triangulation to be unique, we assume the following throughout the paper:

**Assumption 2 (Non-collinearity [14]):** No three points in \( S \) are collinear (i.e., are on the same line).

Note the following implication of the above assumptions: Since we assumed that \( 3 \leq |S| \), that no three points in \( S \) are collinear, and that no four points in \( S \) are co-circular, a Voronoi edge in \( \text{VD}(S) \) can neither be an infinite line, nor can it degenerate into a single point.
D. Proximity graphs

We are now in a position to define the proximity graphs PDT and PuDel, which are the subjects of our study, as well as the two additional proximity graphs GG and UDel which we need in the course of our proofs. We begin with the definitions of GG, UDel and PDT.

Definition 3 (Gabriel graph): The Gabriel graph [4] over the set \( S \), denoted by \( \text{GG}(S) \), joins any two nodes \( u, v \in S \) by an edge iff \( \text{Disk}(u, v) \cap \{u, v\} = \emptyset \).

Definition 4 (Unit Delaunay triangulation): Given a set \( S \), the unit length \( R \), and the corresponding Delaunay triangulation \( \text{Del}(S) \), the Unit Delaunay triangulation \( \text{UDel}(S) \) consists of all edges in \( \text{Del}(S) \) with length at most \( R \), i.e., \( \text{UDel}(S) = \{(u, v) \in \text{Del}(S) : \|uv\| \leq R\} \). We refer to these edges as unit Delaunay edges.

Definition 5 (Partial Delaunay triangulation [8]): For each pair of nodes \( u, v \in S \) let \( w \in S \setminus \{u, v\} \) be one node that maximizes the angle \( \alpha = \angle uvw \), i.e., \( \angle uvw \geq \angle uvv \) for all \( x \in S \setminus \{u, v\} \). \((u, v)\) is an edge in \( \text{PDT}(S) \) if and only if

1. \( \|uv\| \leq R \), and
2. \( (u, v) \in \text{GG}(S) \setminus \text{UDG}(S) \), or
3. \( (u, v, w) \cap \text{N}(u) \setminus \{u, v, w\} = \emptyset \) and \( \sin(\alpha) \geq \|uv\|/R \).

The definition of PuDel at the end of this section requires two auxiliary definitions, which follow next.

Definition 6 (Directed local Delaunay edge): Let \( u, v, w \in S \), s.t. \( \|uv\| \leq R \). The line segment \( \overrightarrow{uv} \) is called directed local Delaunay edge, denoted by \((\overrightarrow{uv}, \overrightarrow{vu})\), if edge \((u, v)\) is contained in \( \text{Del}(\text{N}(u)) \).

Definition 7 (Locally detectable): Let \((\overrightarrow{uv}, \overrightarrow{vu})\) be a directed local Delaunay edge. Edge \((\overrightarrow{uv}, \overrightarrow{vu})\) is said to be locally detectable if and only if there exists a point \( p \in \text{VR}_S(N(u)) \cap \text{VR}_S(N(v)) \) with \( \|up\| \leq R/2 \). We also refer to such a point \( p \) as a witness for the local detectability of a directed local Delaunay edge.

Definition 8 (Partial unit Delaunay graph [13]): The partial unit Delaunay graph of point set \( S \), denoted by \( \text{PuDel}(S) \), contains exactly all locally detectable directed local Delaunay edges.

III. One-Hop Spanner

In this section, we present our simplified version of Xu et al.’s proof [13] that PuDel is a constant stretch spanner of the unit disk graph. We do so in order to assist the reader in following up with our main result presented in Section IV and in order to guarantee its correctness.

The structure of the simplified proof is as follows: We start in proving some basic lemmata. Then, in Theorem 1, Corollaries 1 & 2, and Lemma 6 it is established that \( \text{GG}(S) \cap \text{UDG}(S) \subseteq \text{PuDel}(S) \subseteq \text{UDel}(S) \). From the latter follows immediately that PuDel is a planar graph. Theorem 2 and Corollary 3 concern the one-hop localized computability of PuDel.

Finally, in showing that the direct DT path (see Definition 9) between any two nodes in \( \text{Del}(S) \) is a connected path in \( \text{PuDel}(S) \) (Theorem 3), the proof is given that PuDel is a constant Euclidean stretch spanner of the corresponding unit disk graph (Theorems 4 & 5).

Lemma 1 ([13, Lem. 1]): For any point \( p \in \text{VR}_S(u) \), no nodes from \( S \) are in the interior of \( C_{\|pu\|}(p) \).

Proof\( ^{\star} \): It suffices to consider node \( v \in S \setminus \{u\} \) that is closest to \( p \) (with ties broken arbitrarily). By assumption \( p \in \text{H}(u, v) \). If \( p \notin \text{B}(u, v) \), then it holds that \( \|pu\| < \|pv\| \) and thus \( v \notin C_{\|pu\|}(p) \). If \( p \in \text{B}(u, v) \), then \( \|pu\| = \|pv\| \) and thus, \( v \) is on the boundary of \( C_{\|pu\|}(p) \), but not in the interior of it.

Lemma 2 ([13, Lem. 2]): For a point \( p \notin \text{H}(u, v) \), \( \|pu\| > \|uv\|/2 \) holds.

Proof\( ^{\star} \): Let \( u, v \in S \). By assumption \( p \) is not contained in \( \text{H}(u, v) \), i.e., \( \|pu\| > \|pv\| \). By the triangle inequality it holds that \( \|uv\| \leq \|pu\| + \|pv\| < 2\|pu\| \). Therefore, \( \|uv\|/2 < \|pu\| \).

Lemma 3: \( \text{VR}_S(u) \cap \text{B}(u, v) = \text{VR}_S(v) \cap \text{B}(u, v) \).

Proof: Let \( u, v \in S \). If \( \text{VR}_S(u) \cap \text{B}(u, v) = \emptyset \) then by [14, Prop. V1] it holds that \( \text{VR}_S(v) \cap \text{B}(u, v) = \emptyset \).

If \( \text{VR}_S(u) \cap \text{B}(u, v) \neq \emptyset \), consider any point \( p \) from \( \text{VR}_S(u) \cap \text{B}(u, v) \). With Lemma 1 the interior of \( C_{\|up\|}(p) \) is empty of points from \( S \). Since \( p \in \text{B}(u, v) \), both \( u \) and \( v \) are on the boundary of \( C_{\|up\|}(p) \). Applying [16, Lem. 2.1] yields that \( p \in \text{VR}_S(v) \) and thus, \( p \in \text{VR}_S(v) \cap \text{B}(u, v) \).

Lemma 4 ([13, Lem. 6]): For any node \( u \in S \) it holds that \( \text{VR}_S(u) \subseteq \text{VR}_{\text{N}(u)}(u) \).

Proof\( ^{\star\star} \):

\[
\text{VR}_S(u) = \bigcap_{v \in S \setminus \{u\}} \text{H}(u, v) = \bigcap_{v \in \text{N}(u) \setminus \{u\}} \text{H}(u, v) \cup \bigcup_{v \in S \setminus \text{N}(u)} \text{H}(u, v).
\]

Definition 9 (Direct DT path [17]): Let \( u, v \in S \) and assume for simplicity that \( \overrightarrow{uv} \) is parallel to the x-

\[2\]To indicate the difference between their [13] and our proofs, we introduce the following notation: ‘\( ^{\star} \)’ next to a proof indicates that the proof was missing, ‘\( ^{\star\star} \)’ refers to minor, and ‘\( ^{\star\star\star} \)’ refers to major revisions. The proof for Lemma 4, which we copied from [13] for the sake of completeness, is marked by a ‘\( ^{\star} \)’.  

\[1\]Note that the angle condition is stated here slightly weaker compared to the original definition in [8]. There, it is required that \( \sin(\alpha) > \|uv\|/R \). However, as we define unit disks to be closed disks, this modification does not alter the concept of PDT at all.  

Note: Please ensure that all sections and subsections are properly formatted and that all images and tables are included as required.
axis and that $x(u) < x(v)$. The direct DT path between $u$ and $v$, denoted by $DT(u, v)$, is a sequence $\langle u = b_0, b_1, \ldots, b_{m-1}, b_m = v \rangle$ of nodes from $S$ that corresponds to the sequence of Voronoi regions traversed by walking from $u$ to $v$ along the line segment $\overrightarrow{uv}$ (see Fig. 2 for an illustration). In case a Voronoi edge lies on $\overrightarrow{uv}$, the Voronoi region which lies above $\overrightarrow{uv}$ is chosen. If all nodes $b_i$ along the direct DT path happen to be in the same half-plane defined by the line connecting $u$ and $v$, then this path is said to be one-sided.

**Lemma 5 ([13, Lem. 9]):** Let $u, v \in S$ s.t. the direct DT path from $u$ to $v$ is one-sided, then the length of this path is at most $(\pi/2) \cdot ||uv||$.

Proof**: The proof follows directly from [17, Lem. 1–3]; see further the discussion of [7, Lem. 1].

**Theorem 1 ([13, Cor. 1&2]):** Let $u, v \in S$. Segment $\overrightarrow{uv}$ is a locally detectable directed local Delaunay edge $(u, \vec{v})$ if and only if $(u, v) \in UDel(S)$ and there exists $p \in VR_S(u) \cap VR_S(v)$ with $||up|| \leq R/2$.

Proof****: “$\Rightarrow$”: Let $(u, \vec{v})$ be a locally detectable directed local Delaunay edge. Then the following holds:

1. $||uv|| \leq R$ and $(u, v) \in Del(N(u))$ (by definition of a directed local Delaunay edge).
2. $\exists p \in VR_{N(u)}(u) \cap VR_{N(v)}(v)$ with $||up|| \leq R/2$ (by definition of local detectability), and
3. $p \in VR_{N(u)}(u) \cap B(u, v)$ (follows from 2).

By Lemma 4 it holds that $VR_S(u) \subseteq VR_{N(u)}(u)$. We distinguish two cases:

(i) Assume $p \notin VR_S(u)$: Then there exists $l \in S \setminus N(u)$ s.t. $p \notin H(u, l)$. By Lemma 2 it holds that $||up|| > ||ul||/2$. Because $l \notin N(u)$, $u$ and $l$ are at distance $||ul|| > R$. This implies that $||uv|| > ||ul||/2 > R/2$. But this is a contradiction to the assumption that $||uv|| \leq R/2$ and therefore, we only need to consider case (ii).

(ii) Assume $p \in VR_S(u)$: We know with 3) that $p \in B(u, v)$ and by Lemma 3 it holds that $p \in VR_S(u) \cap VR_S(v)$. By definition of the Delaunay triangulation it follows that $(u, v) \in Del(S)$. With 1) and 2), $||uv|| \leq R$ and $||up|| \leq R/2$ holds. Thus, we obtain that $(u, v)$ is a locally detectable unit Delaunay edge in $UDel(S)$.

“$\Leftarrow$”: Assume $(u, v) \in UDel(S)$. By definition of the Delaunay triangulation we have $VR_S(u) \cap VR_S(v) \neq \emptyset$. Moreover, it holds that $||uv|| \leq R$. Now assume there exists $p \in VR_S(u) \cap VR_S(v)$ with $||up|| \leq R/2$. Note that $p \in B(u, v)$ and therefore, $p \in VR_S(u) \cap B(u, v)$.

By Lemma 4, it holds that $p \in VR_{N(u)}(u)$ and thus, $p \in VR_{N(u)}(u) \cap B(u, v)$. This and Lemma 3 imply that $(\vec{u}, \vec{v})$ is a locally detectable directed local Delaunay edge.

From this theorem and the fact that the unit Delaunay graph $UDel(S)$ is a subgraph of the planar graph $Del(S)$, Corollary 1 directly follows.

**Corollary 1 ([13, Thm. 3]):** $PuDel(S) \subseteq UDel(S)$ and therefore, $PuDel$ is a planar graph.

Proof**: Let $(u, v)$ be any edge in $GG(S) \cap UDG(S)$. We want to show: $(u, v) \in PuDel(S)$.

Let $m$ denote the midpoint of $\overrightarrow{uv}$. Then, $m \in B(u, v)$. By assumption, $||uv|| \leq R$ and there are no nodes from $S$ contained in $Disk(u, v) = C_{\frac{1}{2}||uv||}(m)$. By [18, Thm. 7.4], $m$ is a point on the Voronoi edge between $u$ and $v$ w.r.t. $S$, i.e., $m \in VR_S(u) \cap VR_S(v)$. By Lemma 4, $m \in VR_{N(u)}(u)$ and by Lemma 3 it holds that $m \in VR_{N(v)}(v)$. Thus, $m \in VR_{N(u)}(u) \cap VR_{N(v)}(v)$ and $||um|| \leq R/2$.

**Corollary 2**: If $UDG(S)$ is a connected graph, then $PuDel(S)$ is a connected graph.

Proof**: Bose et al. proved in [1, Lem. 1] that $GG(S) \cap UDG(S)$ is connected if $UDG(S)$ is a connected graph. By Lemma 6, $GG(S) \cap UDG(S)$ is a subgraph of $PuDel(S)$ and therefore it follows immediately that $PuDel(S)$ is connected if $UDG(S)$ is connected.

**Theorem 2 ([13, Thm. 1]):** The partial unit Delaunay graph of a set $S$ is symmetric, i.e., if $(\vec{u}, \vec{v})$ is a locally detectable directed local Delaunay edge, then $(\vec{v}, \vec{u})$ is also a locally detectable directed local Delaunay edge.

Proof**: Let $u, v \in S$ s.t. $(\vec{u}, \vec{v})$ is a locally detectable directed local Delaunay edge. We show: From the view of $u$, $\overrightarrow{uv}$ is also a locally detectable directed local Delaunay edge. I.e., there exists $p \in VR_S(v) \cap VR_{N(u)}(u)$ with $||vp|| \leq R/2$ and $||uv|| \leq R/2$.

By Theorem 1, $(u, v) \in UDel(S)$ and there exists $p \in VR_S(u) \cap VR_S(v)$ with $||vp|| \leq R/2$. Thus, $p \in B(u, v)$ and by definition of directed local Delaunay edges it also holds that $||uv|| \leq R$.

By Lemma 4, $VR_S(v) \subseteq VR_{N(u)}(v)$ and thus, $p \in VR_{N(u)}(v)$. Because $||uv|| \leq R/2$ and $p \in B(u, v)$ it also holds that $||vp|| \leq R/2$.

From this, it follows that $\overrightarrow{uv}$ is a locally detectable directed local Delaunay edge.

With the previous result, we can derive the following:

**Corollary 3 ([13, Thm. 2]):** The partial unit Delaunay graph for the set $S$ can be computed locally by the nodes, using only one-hop neighborhood information.

Proof**: Each node $v \in S$ queries its one-hop neighbors for their positions and computes the local Voronoi diagram $VD(N(v))$. Based upon this diagram, each node decides locally if it holds locally detectable directed local Delaunay edges to its neighbors $u \in S$. The proof follows directly from [17, Lem. 1–3]; see further the discussion of [7, Lem. 1].
If $u$ is incident to $v$, then according to Theorem 2, $u$ can decide that it holds the locally detectable directed local Delaunay edge to $v$, by looking at its one-hop neighborhood.

**Theorem 3 ([13, Cor. 3]):** If $(u, v) \in \text{UDel}(S)$, then the direct DT path from $u$ to $v$ in $\text{Del}(S)$ is a connected path in $\text{PuDel}(S)$.

**Proof:** Let $(u, v) \in \text{UDel}(S)$ and let

$$\text{DT}(u, v) = (u = b_0, b_1, ..., b_{m-1}, b_m = v)$$

denote the direct DT path from $u$ to $v$ in $\text{Del}(S)$. Let $z_i$, $0 \leq i < m$, be the point of intersection of the Voronoi edge between the nodes $b_i, b_{i+1}$ (w.r.t. $S$) with the line segment $\overline{uv}$. I.e., $z_i \in \text{VR}_S(b_i) \cap \text{VR}_S(b_{i+1})$, $z_i \in \text{B}(b_i, b_{i+1})$, and by assumption $z_i \in \overline{uv}$ (see Fig. 2 for an illustration).

As $z_i \in \text{VR}_S(b_i)$, by Lemma 1 it holds that there are no points from $S$ contained in the interior of circle $C_{\|z_i b_i\|}(z_i)$. In particular neither $u$, nor $v$ are inside the circle $C_{\|z_i b_i\|}(z_i)$. Observe that the circle’s center, namely $z_i$, as well as the circle’s diameter, are located on the line segment $\overline{uv}$. By assumption that $(u, v) \in \text{UDel}(S)$ it holds that $\|uv\| \leq R$. It follows that $2 \cdot \|z_i b_i\| \leq \|uv\| \leq R$.

Moreover, since $z_i \in \text{B}(b_i, b_{i+1})$, it follows from the triangle inequality that $\|b_i b_{i+1}\| \leq \|b_i z_i\| + \|z_i b_{i+1}\| = 2 \cdot \|z_i b_i\| \leq R$. Thus, $\|z_i b_i\| \leq R/2$, and $z_i$ is a point of the Voronoi edge between $b_i$ and $b_{i+1}$. From Theorem 1 it then follows that $(b_i, b_{i+1}) \in \text{PuDel}(S)$.

Moreover, this holds for every edge $(b_j, b_{j+1})$, $0 \leq j < m$, along the path $\text{DT}(u, v)$. Therefore, $\text{DT}(u, v)$ is a connected path from $u$ to $v$ in $\text{PuDel}(S)$.

In the following, let $\Pi(u, v)$ denote a connected and acyclic path from $u$ to $v$. The length of the path $\Pi(u, v) = (u = u_1, u_2, ..., u_k = v)$, denoted by $|\Pi(u, v)|$, is defined as follows:

$$|\Pi(u, v)| = \sum_{j=1}^{k-1} \|u_j u_{j+1}\|.$$

**Theorem 4 ([13, Cor. 4]):** For each edge $(u, v)$ from $\text{UDel}(S)$, there exists a connected path $\Pi(u, v)$ in $\text{PuDel}(S)$, where $|\Pi(u, v)| \leq (\pi/2) \cdot \|uv\|$.

**Proof:** Let $(u, v) \in \text{UDel}(S)$ be an arbitrarily chosen but fixed edge. We distinguish two cases:

(i) $\exists p \in \text{VR}_S(u) \cap \text{VR}_S(v)$, where $\|up\| \leq R/2$. In this case $(u, v) \in \text{PuDel}(S)$ according to Theorem 1. The path $\Pi(u, v) := (u, v)$ is a connected path in $\text{PuDel}(S)$ and has length $|\Pi(u, v)| = \|uv\| < \pi/2 \cdot \|uv\|$.

(ii) $\not\exists p \in \text{VR}_S(u) \cap \text{VR}_S(v)$, where $\|up\| \leq R/2$. Let $q$ be any point from the intersection of the Voronoi regions of $u$ and $v$ w.r.t. $S$, i.e., $q \in \text{VR}_S(u) \cap \text{VR}_S(v)$ (see Fig. 3 for an illustration). Because $q \in \text{VR}_S(u)$ and $q \in \text{VR}_S(v)$ it holds that $q \in \text{VR}_S(u) \cap \text{B}(u, v)$. Thus, and by assumption we get that $\|uq\| = \|vq\| > R/2$. Denote the midpoint of the line segment $\overline{uv}$ by $m$. Since $(u, v) \in \text{UDel}(S)$, $\|uv\| \leq R$ and $\|um\| = \|vm\| \leq \|uq\|$. In particular, this implies that $m \neq q$. Now consider the following two circles with center $m$ and $q$, respectively, both of which contain $u$ and $v$ on their boundaries:

- $\text{Disk}(u, v)$ is the circle with center $m$, radius $\|um\|$, and has line segment $\overline{uv}$ as its diameter.
- $C_{\|uq\|}(q)$ is the circle with center $q$, radius $\|uq\|$, and contains $u$ and $v$ on its boundary because $\|uq\| = \|vq\|$.

Note that these circles are different, as their centers as well as their radii differ strictly. Moreover, observe that one of the semicircles of $\text{Disk}(u, v)$ that is bounded by $\overline{uv}$, which we will call $A$ hereafter, is completely contained by $C_{\|uq\|}(q)$. Since $q \in \text{VR}_S(u)$, by Lemma 1 it holds that no points from $S$ are in the interior of circle $C_{\|uq\|}(q)$. From this, we can conclude that no points from $S \setminus \{u, v\}$ are contained inside, or on the boundary of semicircle $A$.

Let $\text{DT}(u, v)$ denote the direct DT path from $u$ to $v$ in $\text{Del}(S)$. From [17, Lem. 2] we know that all nodes along the path $\text{DT}(u, v)$ are contained within, or on the boundary of $\text{Disk}(u, v)$. But since semicircle $A$ of $\text{Disk}(u, v)$ is empty of nodes from $S \setminus \{u, v\}$, all remaining nodes from $\text{DT}(u, v)$ must be contained in the remaining semicircle $\text{Disk}(u, v) - A$. I.e., all points from $\text{DT}(u, v)$ are contained in the half-plane that has $\overline{uv}$ as its boundary and contains $\text{Disk}(u, v) - A$. Thus, $\text{DT}(u, v)$ is one-sided. Moreover, by Theorem 3, $\text{DT}(u, v)$ is a connected path in $\text{PuDel}(S)$. Thus, for $\Pi(u, v) := \text{DT}(u, v)$ it holds by Lemma 5 that $|\Pi(u, v)| \leq \pi/2 \cdot \|uv\|$.
claim (see [7, Thm. 6] and [9, Lem. 1]) concerning the spanning ratio of the unit Delaunay triangulation.

Theorem 5 ([13, Thm. 4]): Let $S$ be a set of nodes s.t. UDG($S$) is a connected graph. Then, PuDel($S$) is an Euclidean $1 + \sqrt{5}/4 \pi^2$-spanner of UDG($S$).

Proof**: Let $S$ be a set of nodes such that UDG($S$) is connected. According to [7, Thm. 5], the unit Delaunay triangulation UDel($S$) is an Euclidean $1 + \sqrt{5}/4 \pi^2$-spanner of UDG($S$).

Next it is shown that PuDel($S$) is a $\pi/2$-spanner of UDel($S$) from which the theorem then follows.

Let $u, v \in S$ be two arbitrarily chosen but fixed nodes. Assume the shortest path from $u$ to $v$ in UDel($S$) is given by

$$\Pi_{UDel}(u, v) = (u = u_0, u_1, ..., u_k = v).$$

By Theorem 4, for each edge $(x, y) \in UDel(S)$ there exists a connected path from $x$ to $y$ in PuDel($S$), whose length is smaller than or equal to $||uv|| \cdot ||uv||$. In the following, we denote this path by $x \Rightarrow y$. By assumption, each pair $(u_i, u_{i+1})$, for $0 \leq i < k$, represents an edge in UDel($S$). Thus, the concatenation of all such paths $u_i \Rightarrow u_{i+1}$, for $0 \leq i < k$, is a connected path from $u$ to $v$ in PuDel($S$).

Let $\Pi_{PuDel}(u, v)$ denote the shortest path from $u$ to $v$ in PuDel($S$). The length of $\Pi_{PuDel}(u, v)$ is as follows:

$$||\Pi_{PuDel}(u, v)|| \leq \sum_{i=0}^{k-1} ||u_i \Rightarrow u_{i+1}|| \leq \sum_{i=0}^{k-1} \frac{\pi}{2} ||u_i u_{i+1}|| = \frac{\pi}{2} \sum_{i=0}^{k-1} ||u_i u_{i+1}|| = \frac{\pi}{2} \Pi_{UDel}(u, v).$$

IV. EQUIVALENCE OF PuDEL AND PDT

In this section, our main result is presented: We show that the partial unit Delaunay triangulation (PuDel, introduced in [13]) is equivalent to the partial Delaunay Triangulation (PDT, introduced in [8]). In showing the equivalence, we can positively answer the open question (see e.g., [11]) if PDT is a constant Euclidean stretch spanner of the unit disk graph.

The proof’s structure is as follows: First we show three lemmata, which serve proving Theorem 6, our main result. To prove the latter, we first show the more involved case that $(u, v) \in PuDel(S) \Rightarrow (u, v) \in PDT(S)$. Afterwards, we prove the remaining implication, namely that $(u, v) \in PDT(S) \Rightarrow (u, v) \in PuDel(S)$.

Lemma 7: Let $(u, v) \in PuDel(S)$ and let $p \in \mathbb{R}^2$ be any witness for the local detectability of $(\overrightarrow{uv}, \overrightarrow{wp})$. It holds that $p$ is also a witness for the local detectability of $(\overrightarrow{uv})$. Proof: Assume $p$ is a witness for the local detectability of $(\overrightarrow{uv}, \overrightarrow{wp})$. Then, there must exist $l \in H(u, l)$ such that $p \notin H(v, l)$. Thus, $||lp|| < ||vp||$. From this, we can conclude as follows that $l$ is a one-hop neighbor of $u$:

$$||ul|| \leq ||uv|| + ||lp|| < ||uv|| + ||vp|| \leq R.$$ 

Moreover, because $||lp|| < ||vp|| = ||vp||$ it holds that $p \notin H(u, l)$. But this contradicts the initial assumption that $p$ is a witness for the local detectability of $(\overrightarrow{uv}, \overrightarrow{wp})$. Thus, $p$ must be a witness for the local detectability of $(\overrightarrow{uv})$ as well.

Definition 10 (Common local Voronoi edge): For two neighboring nodes $u, v \in S$, we define the common local Voronoi edge of $u$ and $v$ w.r.t. their one-hop neighborhoods, denoted by CLVE($u, v$), as follows:

$$CLVE(u, v) = \{VR_N(u) \cap VR_N(v)\} \cap \{VR_{N(v)}(u) \cap VR_{N(u)}(v)\}.$$

For an example, see Fig. 5, where CLVE($u, v$) is represented as the bold printed line segment $\overrightarrow{ab}$.

Lemma 8: Let $u, v \in S$. If $CLVE(u, v) \neq \emptyset$, then $|CLVE(u, v)| > 1$. 

Proof: We prove this lemma by contradiction. Assume there are nodes $u, v \in S$ such that $CLVE(u, v) \neq \emptyset$ and $|CLVE(u, v)| = 1$. We distinguish two cases:

1) $|VR_N(u) \cap VR_N(v)| = 1$ or $|VR_N(v) \cap VR_N(u)| = 1$;
2) $|VR_N(u) \cap VR_N(v)| > 1$ and $|VR_N(v) \cap VR_N(u)| > 1$.

Case 1): At least one of the two local Voronoi diagrams is degenerated. This contradicts the assumption that no four points in $S$ are co-circular.
We are now in a position to present our main result, namely the proof of equivalence of PuDel and PDT.

Theorem 6: Let $S \subset \mathbb{R}^2$ be a finite set of nodes of size $|S| \geq 3$, which satisfies the non-degeneracy and the non-collinearity assumptions. For any two nodes $u, v \in S$ the following holds: $(u, v) \in \text{PuDel}(S)$ if and only if $(u, v) \in \text{PDT}(S)$.

Proof: We show the equivalence in two steps:

(1) $(u, v) \in \text{PuDel}(S) \Rightarrow (u, v) \in \text{PDT}(S)$, and
(2) $(u, v) \in \text{PDT}(S) \Rightarrow (u, v) \in \text{PuDel}(S)$.

W.l.o.g. we assume for the remainder of this proof that line segment $\overline{uv}$ is parallel to the x-axis. Moreover, let $m$ always denote the midpoint of $\overline{uv}$ and $R$ the unit length.

In addition, note the following: If $(u, v) \in \text{UDG}(S)$, then $\text{Disk}(u, v)$ is entirely contained by the unit disk centered at $u$. Thus, $\text{Disk}(u, v) \cap \left( N(u) \setminus \{u, v\} \right) = \emptyset$ if and only if $\text{Disk}(u, v) \cap \left( S \setminus \{u, v\} \right) = \emptyset$.

(1) $(u, v) \in \text{PuDel}(S) \Rightarrow (u, v) \in \text{PDT}(S)$

Recall that an edge $(x, y)$ is contained in PDT(S) if $\|xy\| \leq R$ and if either $(x, y) \in \text{GG}(S) \cap \text{UDG}(S)$, or there exists a circle $C(x, z, y)$ being empty of nodes from $N(x) \setminus \{x, y, z\}$ and $\sin(\angle xzy) \geq \|xy\|/R$.

In the case under consideration, $\|uv\| \leq R$ holds trivially, because $(u, v)$ is assumed to be an edge in PuDel(S). Moreover, we can derive the following facts from $(u, v)$ being an edge in PuDel(S): $(u, \hat{v})$ and $(\hat{v}, a)$ are locally detectable direct local Delaunay edges, i.e., there exists a point $p \in \text{VR}_{N(u)}(u) \cap \text{VR}_{N(v)}(v)$ with $\|up\| = \|vp\| \leq R/2$.

Moreover, $p \in \text{VR}_{N(u)}(v) \cap \text{VR}_{N(v)}(u)$ by Lemma 7. From this and from Lemma 8, it follows in turn that $|\text{CLVE}(u, v)| > 1$. Note in addition that it cannot be the case that $\text{CLVE}(u, v)$ is equivalent to the straight line $B(u, v)$, as this would contradict to the non-degeneracy assumption (for details see Assumptions 1 & 2 and their implications, discussed in Section II-C).

Thus, in the case under consideration CLVE$(u, v)$ can exclusively be of the following two types: It is either (a) a line segment $[a, b] \subset B(u, v)$, where $a \neq b$, or (b) it is a half-line lying on $B(u, v)$.

Consider case (a), where CLVE$(u, v)$ is a line segment $[a, b] \subset B(u, v)$, as illustrated in Fig. 5. Recall that $|\text{CLVE}(u, v)| > 1$, and therefore $a \neq b$ holds.

W.l.o.g. we assume that $y(a) > y(b)$ and $\|ma\| \leq \|mb\|$ (the case where $\|ma\| \geq \|mb\|$ can be proven analogously). Observe that the latter assumption implies, that $y(m) > y(b)$, i.e., $b$ lies (strictly) below $\overline{uv}$.

If $y(a) > y(m)$, then there exists $[a, b] \subset \text{CLVE}(u, v)$ with $m \in [a, b]$ and $m \neq a, b$. It follows immediately from Lemma 9 that $(u, v) \in \text{GG}(S) \cap \text{UDG}(S)$ and in turn that $(u, v) \in \text{PDT}(S)$.
Therefore, let us assume \( y(a) \leq y(m) \). Then, either \( a = m \), or \( a \) lies strictly below \( \overline{vw} \) but within \( C_{R/2}(u) \cap C_{R/2}(v) \), as \( (u, v) \in \text{PuDel}(S) \).

Observe the following: \( a \) is a Voronoi vertex w.r.t. either \( \text{VD}(N(u)) \), or \( \text{VD}(N(v)) \). Let us assume the former holds. Then, in accordance with [19, Thm. 5.7], there must exist a node \( l \) s.t. \( a \) is the point of intersection of \( \text{VR}_{N(u)}(u) \), \( \text{VR}_{N(u)}(v) \), and \( \text{VR}_{N(u)}(l) \). But because of the position of \( a \), node must be contained in \( \text{Disk}(u, v) \). Therefore, \( l \in (N(u) \cap N(v)) \) and \( l \) is a Voronoi vertex w.r.t. \( \text{VD}(N(v)) \) as well. From [19, Thm. 5.8] it now follows that \( C(u, v, l) \) is empty of nodes from \( (N(u) \cup N(v)) \setminus \{u, v, l\} \). For the remainder let \( \alpha = \angle ulv \).

Assume \( a = m \), then \( l \) must be located on the boundary of \( \text{Disk}(u, v) \). Hence, \( \text{Disk}(u, v) \) and \( C(u, v, l) \) coincide and \( \text{Disk}(u, v) \) must also be empty of nodes from \( (N(u) \cup N(v)) \setminus \{u, v, l\} \). Moreover, \( \alpha = \pi/2 \) (by Thales’ theorem). Since \( \|uv\| \leq R \), \( \sin(\alpha) = \frac{1}{R} \) holds as well. From this, we can conclude that \((u, v) \in \text{PDT}(S)\) holds.

Let us now assume that both points \( a \) and \( b \) are located strictly below \( \overline{vw} \). Because \((\overline{w}, \overline{b})\) is locally detectable and because of the assumption that \( y(a) > y(b) \), \( a \in C_{R/2}(u) \), whereas \( b \) may, or may not be contained in \( C_{R/2}(u) \). In this case \( l \) must be contained in the interior of \( \text{Disk}(u, v) \). Recall that \((u, v, l)\) with center \( a \) is empty of nodes from \( (N(u) \cup N(v)) \setminus \{u, v, l\} \). If we can now show that \( \sin(\alpha) \geq \frac{\|uv\|}{R} \), for all possible positions of \( a \), then it follows that \((u, v) \in \text{PDT}(S)\) holds.

Suppose we move \( a \) along \( \text{CLVE}(u, v) \) towards the lower point of intersection of \( C_{R/2}(u) \) and \( C_{R/2}(v) \), denoted by \( j \). It can be observed that with monotonically decreasing value \( y(a) \), angle \( \alpha \) increases monotonically, whereas \( \sin(\alpha) \) decreases monotonically (because \( \pi/2 < \alpha < \pi \)). Moreover, \( \alpha \) reaches its maximum and \( \sin(\alpha) \) reaches its minimum, if \( a = j \), as we are only considering cases where \((\overline{w}, \overline{b})\) remains locally detectable, i.e., cases where \( \|ua\| \leq R/2 \).

Assume \( a = j \) and treat \( \overline{vw} \) as a chord in \( C(u, v, l) \). By elementary geometry, we get the following:

\[
\sin(\alpha) = \frac{\|uv\|}{2 \cdot \|ua\|} = \frac{\|uv\|}{2 \cdot R/2} = \frac{\|uv\|}{R}.
\]

Thus, \( \sin(\alpha) \geq \frac{\|uv\|}{R} \) for all possible positions of \( a \) under consideration. We conclude that \((u, v) \in \text{PDT}(S)\).

Case (b), where \( \text{CLVE}(u, v) \) is a half-line, is a special case of (a) and can be proven equivalently: Let \( p \) denote the unique endpoint of \( \text{CLVE}(u, v) \) and apply the same line of argumentation as used in the proof of (a) on point \( p \), instead of point \( a \). Then, \((u, v) \in \text{PDT}(S)\) follows for this case as well. Thus, the first part of this theorem holds and we can proceed with the remaining implication.

\( (2) \ (u, v) \in \text{PDT}(S) \Rightarrow (u, v) \in \text{PuDel}(S) \)

For the remainder of this proof let \((u, v) \in \text{PDT}(S) \). Recall that we assumed \( \overline{vw} \) to be parallel to the \( x \)-axis. Let \( l \) be the angle maximizing node w.r.t. \( \overline{vw} \), i.e., \( \angle ulv \geq \angle uv \cdot l \in (N(u) \cup N(v)) \setminus \{u, v\} \) (breaking ties arbitrarily) and let \( \alpha = \angle ulv \) (see Fig. 6 for an illustration).

According to the definition of PDT, \( \|uv\| \leq R \) and \((u, v) \) is either a Gabriel edge, or circle \( C(u, v, l) \) is empty of nodes from \( N(u) \setminus \{u, v, l\} \) and \( \sin(\alpha) \geq \frac{\|uv\|}{R} \) holds.

If \((u, v) \) is a Gabriel edge, then \((u, v) \in \text{PuDel}(S) \) because \( \text{GG}(S) \cap \text{UDG}(S) \subseteq \text{PuDel}(S) \) (by Lemma 6). Therefore, assume that \((u, v) \notin \text{GG}(S) \). In order to prove that \((u, v) \in \text{PuDel}(S) \) it suffices to show that \( \overline{vw} \) is a locally detectable directed local Delaunay edge. I.e., we need to show:

(a) \( \text{VR}_{N(u)}(u) \cap \text{VR}_{N(u)}(v) \neq \emptyset \) (i.e., \( \overline{vw} \) is a directed local Delaunay edge),

(b) \( \exists p \in \mathbb{R}^2 : p \in \text{VR}_{N(u)}(u) \cap \text{VR}_{N(u)}(v) \), where \( \|up\| \leq R/2 \) (the directed local Delaunay edge is locally detectable).

Obviously, (b) implies (a) and it suffices to show (b).

By assumption \((u, v) \notin \text{GG}(S) \), that is, \( l \) is either on the boundary of, or in the interior of \( \text{Disk}(u, v) \). Therefore, \( \|ul\| < \|uv\| \) which implies that \( l \in N(u) \). Let \( p \) denote the midpoint of circle \( C(u, v, l) \) and observe that \( p \in \text{B}(u, v) \) as shown in Fig. 6. Because \((u, v) \in \text{PDT}(S) \) and \( l \) is assumed to be angle maximizing w.r.t. \( \overline{vw} \), the interior of \( C(u, v, l) \) is empty of nodes from \( N(u) \setminus \{u, v, l\} \) and \( C(u, v, l) \) is entirely contained by the unit disk centered at \( u \). Thus, \( C(u, v, l) \) is empty of nodes from \( S \setminus \{u, v, l\} \). From [16, Lemma 2.1] it follows that \( p \) is a Voronoi vertex in \( \text{VD}(S) \) joining the Voronoi regions \( \text{VR}_S(u) \), \( \text{VR}_S(v) \), and \( \text{VR}_S(l) \). By Lemma 4 \( p \in \text{VR}_{N(u)}(u) \cap \text{VR}_{N(u)}(v) \) holds as well. It remains to show that \( p \) is a witness for \((\overline{w}, \overline{b})\)’s local detectability for all possible positions of \( l \).

If \( l \) lies on the boundary of \( \text{Disk}(u, v) \), then \( p = m \). Since \( \|uv\| \leq R \), \( \|up\| \leq R/2 \) holds.

Therefore, assume \( l \) lies strictly inside \( \text{Disk}(u, v) \). W.l.o.g. assume \( m \) is contained in the

![Fig. 6. Illustration for the proof of Theorem 6, part (2): the dashed line represents \( \text{VD}(N(u)) \)](https://example.com/fig6.png)
upper half-circle. Because \((u,v) \in \text{PDT}(S)\), \(\sin(\alpha) \geq \|uv\|/R\) holds. Observe that with increasing value of \(\alpha\), which is the case if the distance between \(l\) and the midpoint \(m\) of \(uv\) decreases, the difference between \(\sin(\alpha)\) and \(\|uv\|/R\) decreases. Moreover, with increasing angle \(\alpha\), the radius of \(C(u,v,l)\) is monotonically increasing and so is the distance between \(u\) and \(p\). Obviously, while \((u,v)\) is an edge in \(\text{PDT}(S)\), \(\|up\|\) is maximized, when \(\sin(\alpha) = \|uv\|/R\). Therefore, it suffices to show that \(\|up\| \leq R/2\) for the latter case. Let \(r\) denote the radius of \(C(u,v,l)\). The law of sines yields the following:

\[ R = \frac{\|uv\|}{\sin(\alpha)} = 2 \cdot r = 2 \cdot \|up\|, \]

I.e., independent of \(l\)'s position, while \((u,v) \in \text{PDT}(S)\) holds, \(p\) is a witness for the local detectability of the directed local Delaunay edge \((u,v)\).

V. Conclusion

In this paper, we proved the equivalence of PuDel [13] and PDT [8]. Our motivation for this proof is that PuDel is a planar spanner for the unit disk graph, and thus, PDT is one as well. The latter has been an open question for years.

Both PuDel and PDT are constructed by one-hop localized algorithms. Before spanning properties of PuDel and PDT were known, localized planar spanner construction had been tied to two-hop message exchange. Two-hop message exchange, however, means extra protocol overhead. In contrast, algorithms based on one-hop neighbor information can often get this information for free. For instance, the MAC layer often requires regular beacon intervals anyhow; which makes each node implicitly aware of its one-hop neighbors. We know now that planar spanners can be constructed almost for free.

This seminal insight has to be acknowledged to the authors of PuDel [13]. However, their spanning property proof suffers from omitted significant proof steps and we had to revise their proof here to assure validity of their and our result.

There also exists a two-hop variant of PDT, namely PDT2. Since PDT is a subgraph of PDT2 [20], we know now that PDT2 is a planar spanner as well. Moreover, the PDT approach has found several applications (e.g., [21], [22], [23]) because the resulting subgraph is at least as dense as the corresponding Gabriel graph. In proving PDT to be a planar spanner, we can directly issue theoretical guarantees to these applications.

Further research should mainly focus on the incorporation of the PDT approach as default topology control technique to the well-known localized geographic routing algorithms. This will result in improved message efficiency of these algorithms and in turn, will lead to improved energy efficient communication in wireless ad hoc and sensor networks. Also, it should be investigated if the worst-case upper bound on PDT's stretch factor can be improved further, and whether or not PDT2 provides a better worst-case stretch factor than the localized Delaunay graph [8].

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